Graph Algorithms

Graphs are useful for representing a variety of concepts:
- Data Structures
- Relationships
- Families
- Communication Networks
- Road Maps

A Tree Proof

- **Definition:** A free tree is a connected, undirected graph that has no cycles.
- **Theorem:** If \( T \) is a free tree having \( n \) vertices, then \( T \) has exactly \( n - 1 \) edges.
- **Proof:** By induction on \( n \).
- **Base Case:** \( n = 1 \). \( T \) consists of 1 vertex and 0 edges.
- **Inductive Hypothesis:** The theorem is true for a tree having \( n - 1 \) vertices.
- **Inductive Step:**
  - If \( T \) has \( n \) vertices, then \( T \) contains a vertex of degree 1.
  - Removing that vertex and its incident edge to obtain \( T' \), a free tree with \( n - 1 \) vertices.
  - By IH, \( T' \) has \( n - 2 \) edges.
  - Thus, \( T \) has \( n - 1 \) edges.

Graph Traversals

Various problems require a way to traverse a graph — that is, visit each vertex and edge in a systematic way.

Three common traversals:
- **Eulerian tours**
  - Traverse each edge exactly once
- **Depth-first search**
  - Keeps vertices on a stack
- **Breadth-first search**
  - Keeps vertices on a queue

A graph \( G = (V, E) \) consists of a set of vertices \( V \), and a set of edges \( E \), such that each edge in \( E \) is a connection between a pair of vertices in \( V \).
- Directed vs. Undirected
- Labeled graph, weighted graph
- Labels for edges vs. weights for edges
- Multiple edges, loops
- Cycle, Circuit, path, simple path, tours
- Bipartite, acyclic, connected
- Rooted tree, unrooted tree, free tree

This is close to a satisfactory definition for free tree. There are several equivalent definitions for free trees, with similar proofs to relate them.

Why do we know that some vertex has degree 1? Because the definition says that the Free Tree has no cycles.
Eulerian Tours

A circuit that contains every edge exactly once.

Example:

Tour: b a f c d e.

Example:

No Eulerian tour. How can you tell for sure?

Eulerian Tour Proof

- **Theorem**: A connected, undirected graph with \( m \) edges that has no vertices of odd degree has an Eulerian tour.
- **Proof**: By induction on \( m \).
- **Base Case**:
- **Inductive Hypothesis**:
- **Inductive Step**:
  - Start with an arbitrary vertex and follow a path until you return to the vertex.
  - Remove this circuit. What remains are connected components \( G_1, G_2, \ldots, G_k \) each with nodes of even degree and \(< m \) edges.
  - By IH, each connected component has an Eulerian tour.
  - Combine the tours to get a tour of the entire graph.

Depth First Search

```java
void DFS(Graph G, int v) { // Depth first search
    PreVisit(G, v); // Take appropriate action
    G.setMark(v, VISITED);
    for (Edge w = each neighbor of v)
        if (G.getMark(G.v2(w)) == UNVISITED)
            DFS(G, G.v2(w));
    PostVisit(G, v); // Take appropriate action
}
```

Initial call: \( \text{DFS}(G, r) \) where \( r \) is the root of the DFS.

Cost: \( \Theta(|V| + |E|) \).

Depth First Search Example

The directions are imposed by the traversal. This is the Depth First Search Tree.
DFS Tree

If we number the vertices in the order that they are marked, we get DFS numbers.

**Lemma 7.2:** Every edge \( e \in E \) is either in the DFS tree \( T \), or connects two vertices of \( G \), one of which is an ancestor of the other in \( T \).

**Proof:** Consider the first time an edge \( (v, w) \) is examined, with \( v \) the current vertex.
- If \( w \) is unmarked, then \( (v, w) \) is in \( T \).
- If \( w \) is marked, then \( w \) has a smaller DFS number than \( v \) AND \( (v, w) \) is an unexamined edge of \( w \).
- Thus, \( w \) is still on the stack. That is, \( w \) is on a path from \( v \).

DFS for Directed Graphs

- Main problem: A connected graph may not give a single DFS tree.
- Forward edges: \( (1, 3) \)
- Back edges: \( (5, 1) \)
- Cross edges: \( (6, 1), (8, 7), (9, 5), (9, 8), (4, 2) \)
- **Solution:** Maintain a list of unmarked vertices.
  - Whenever one DFS tree is complete, choose an arbitrary unmarked vertex as the root for a new tree.

Directed Cycles

**Lemma 7.4:** Let \( G \) be a directed graph. \( G \) has a directed cycle iff every DFS of \( G \) produces a back edge.

**Proof:**
- Suppose a DFS produces a back edge \( (v, w) \).
  - \( v \) and \( w \) are in the same DFS tree, \( w \) an ancestor of \( v \).
  - \( (v, w) \) and the path in the tree from \( w \) to \( v \) form a directed cycle.
- Suppose \( G \) has a directed cycle \( C \).
  - Do a DFS on \( G \).
  - Let \( w \) be the vertex of \( C \) with smallest DFS number.
  - Let \( (v, w) \) be the edge of \( C \) coming into \( w \).
  - \( v \) is a descendant of \( w \) in a DFS tree.
  - Therefore, \((v, w)\) is a back edge.

Breadth First Search

- Like DFS, but replace stack with a queue.
- Visit vertex's neighbors before going deeper in tree.
Breadth First Search Algorithm

void BFS(Graph G, int start) {
    Queue Q(G.n());
    Q.enqueue(start);
    G.setMark(start, VISITED);
    while (!Q.isEmpty()) {
        int v = Q.dequeue(); // Take appropriate action
        for (Edge w = each neighbor of v) {
            if (G.getMark(G.v2(w)) == UNVISITED) {
                G.setMark(G.v2(w), VISITED);
                Q.enqueue(G.v2(w));
            }
        }
        PostVisit(G, v); // Take appropriate action
    }
}

Topological Sort

Problem: Given a set of jobs, courses, etc. with prerequisite constraints, output the jobs in an order that does not violate any of the prerequisites.

Topological Sort Algorithm

void toposort(Graph G) { // Top sort: recursive
    for (int i=0; i<G.n(); i++) // Initialize Mark
        G.setMark(i, UNVISITED);
    for (i=0; i<G.n(); i++) // Process vertices
        if (G.getMark(i) == UNVISITED) {
            tophelp(G, i); // Call helper
        }
}

void tophelp(Graph G, int v) { // Helper function
    G.setMark(v, VISITED);
    for (Edge w = each neighbor of v) {
        if (G.getMark(G.v2(w)) == UNVISITED) {
            tophelp(G, G.v2(w));
        }
    }
    printout(v); // PostVisit for Vertex v
}

Prints in reverse order.
Queue-based Topological Sort

void topsort(Graph G) { // Top sort: Queue
    Queue Q(G.n()); int Count[G.n()];
    for (int v=0; v<G.n(); v++) Count[v] = 0;
    for (v=0; v<G.n(); v++) // Process every edge
        for (Edge w each neighbor of v)
            Count[G.v2(w)]++; // Add to v2’s count
    for (v=0; v<G.n(); v++) // Initialize Queue
        if (Count[v] == 0) Q.enqueue(v);
    while (!Q.isEmpty()) { // Process the vertices
        int v = Q.dequeue(); // PreVisit for v
        printout(v);
        for (Edge w = each neighbor of v) {
            Count[G.v2(w)]--; // One less prereq
            if (Count[G.v2(w)]==0) Q.enqueue(G.v2(w));
        }
    }
}

Shortest Paths Problems

Input: A graph with weights or costs associated with each edge.

Output: The list of edges forming the shortest path.

Sample problems:
- Find the shortest path between two specified vertices.
- Find the shortest path from vertex S to all other vertices.
- Find the shortest path between all pairs of vertices.

Our algorithms will actually calculate only distances.

Shortest Paths Definitions

d(A, B) is the shortest distance from vertex A to B.

w(A, B) is the weight of the edge connecting A to B.
- If there is no such edge, then w(A, B) = \infty.

Single Source Shortest Paths

Given start vertex s, find the shortest path from s to all other vertices.

Try 1: Visit all vertices in some order, compute shortest paths for all vertices seen so far, then add the shortest path to next vertex x.

Problem: Shortest path to a vertex already processed might go through x.
Solution: Process vertices in order of distance from s.
**Dijkstra's Algorithm Example**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>10</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td></td>
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<td>5</td>
<td>2</td>
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<td></td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

**Dijkstra's Algorithm: Array (1)**

```c
void Dijkstra(Graph G, int s) { // Use array
    int D[G.n()];
    for (int i=0; i<G.n(); i++) // Initialize
        D[i] = INFINITY;
    D[s] = 0;
    for (i=0; i<G.n(); i++) { // Process vertices
        int v = minVertex(G, D);
        if (D[v] == INFINITY) return; // Unreachable
        G.setMark(v, VISITED);
        for (Edge w = each neighbor of v)
            if (D[G.v2(w)] > (D[v] + G.weight(w)))
                D[G.v2(w)] = D[v] + G.weight(w);
    }
}
```

**Dijkstra's Algorithm: Array (2)**

```c
class Elem { public: int vertex, dist; };
int key(Elem x) { return x.dist; }
void Dijkstra(Graph G, int s) { // priority queue
    int v; Elem temp;
    int D[G.n()]; Elem E[G.e()];
    temp.dist = 0; temp.vertex = s; E[0] = temp;
    heap H(E, 1, G.e()); // Create the heap
    for (int i=0; i<G.n(); i++) D[i] = INFINITY;
    D[s] = 0;
    for (i=0; i<G.n(); i++) { // Get distances
        do { temp = H.removemin(); v = temp.vertex; }
            while (G.getMark(v) == VISITED);
        G.setMark(v, VISITED);
        if (D[v] == INFINITY) return; // Unreachable
    }
}
```

**Dijkstra's Algorithm: Priority Queue (1)**

```c
class Elem { public: int vertex, dist; };
int key(Elem x) { return x.dist; }
void Dijkstra(Graph G, int s) { // priority queue
    Elem temp;
    int D[G.n()]; Elem E[G.e()];
    temp.dist = 0; temp.vertex = s; E[0] = temp;
    heap H(E, 1, G.e()); // Create the heap
    for (int i=0; i<G.n(); i++) D[i] = INFINITY;
    D[s] = 0;
    for (i=0; i<G.n(); i++) { // Get distances
        do { temp = H.removemin(); v = temp.vertex; }
            while (G.getMark(v) == VISITED);
        G.setMark(v, VISITED);
        if (D[v] == INFINITY) return; // Unreachable
    }
}
```

Approach 1: Scan the table on each pass for closest vertex. Total cost: $\Theta(|V|^2 + |E|) = \Theta(|V|^2)$.
Dijkstra's Algorithm: Priority Queue (2)

for (Edge w = each neighbor of v)
    if (D[G.v2(w)] > (D[v] + G.weight(w))) {
        D[G.v2(w)] = D[v] + G.weight(w);
        temp.dist = D[G.v2(w)];
        temp.vertex = G.v2(w);
        H.insert(temp); // Insert new distance
    }

• Approach 2: Store unprocessed vertices using a min-heap to implement a priority queue ordered by D value. Must update priority queue for each edge.
• Total cost: Θ((|V| + |E|) log |V|).

All Pairs Shortest Paths

• For every vertex u, v ∈ V, calculate d(u, v).
• Could run Dijkstra's Algorithm |V| times.
• Better is Floyd's Algorithm.
• Define a k-path from u to v to be any path whose intermediate vertices all have indices less than k.

Floyd's Algorithm

void Floyd(Graph G) { // All-pairs shortest paths
    int D[G.n()][G.n()]; // Store distances
    for (int i=0; i<G.n(); i++) // Initialize D
        for (int j=0; j<G.n(); j++)
            D[i][j] = G.weight(i, j);
    for (int k=0; k<G.n(); k++) // Compute k paths
        for (int i=0; i<G.n(); i++)
            for (int j=0; j<G.n(); j++)
                if (D[i][j] > (D[i][k] + D[k][j]))
                    D[i][j] = D[i][k] + D[k][j];
}

Minimum Cost Spanning Trees

Minimum Cost Spanning Tree (MST) Problem:
• Input: An undirected, connected graph G.
• Output: The subgraph of G that
  • has minimum total cost as measured by summing the values for all of the edges in the subset, and
  • keeps the vertices connected.
**Key Theorem for MST**

Let $V_1$, $V_2$ be an arbitrary, non-trivial partition of $V$. Let $(v_1, v_2), v_1 \in V_1, v_2 \in V_2$, be the cheapest edge between $V_1$ and $V_2$. Then $(v_1, v_2)$ is in some MST of $G$.

**Proof:**

- Let $T$ be an arbitrary MST of $G$.
- If $(v_1, v_2)$ is in $T$, then we are done.
- Otherwise, adding $(v_1, v_2)$ to $T$ creates a cycle $C$.
- At least one edge $(u_1, u_2)$ of $C$ other than $(v_1, v_2)$ must be between $V_1$ and $V_2$.
- $c(u_1, u_2) \geq c(v_1, v_2)$.
- Let $T' = T \cup \{(v_1, v_2)\} - \{(u_1, u_2)\}$.
- Then, $T'$ is a spanning tree of $G$ and $c(T') \leq c(T)$.
- But $c(T)$ is minimum cost.

Therefore, $c(T') = c(T)$ and $T'$ is a MST containing $(v_1, v_2)$.

**Key Theorem Figure**

[Diagram showing a graph with marked and unmarked vertices, Prim's edge, and cycle creation]

**Prim's MST Algorithm (1)**

```c
void Prim(Graph G, int s) { // Prim's MST alg
    int D[G.n()]; int V[G.n()]; // Distances
    for (int i=0; i<G.n(); i++) { // Initialize
        D[i] = INFINITY;
        D[s] = 0;
    }
    for (int i=0; i<G.n(); i++) // Process vertices
    {
        int v = minVertex(G, D);
        G.setMark(v, VISITED);
        if (v != s) AddEdgeToMST(V[v], v);
        if (D[v] == INFINITY) return; //v unreachable
        for (Edge w = each neighbor of v)
            if (D[G.v2(w)] > G.weight(w)) {
                D[G.v2(w)] = G.weight(w); // Update dist
                V[G.v2(w)] = v; // who came from
            }
    }
}
```

**Prim's MST Algorithm (2)**

```c
int minVertex(Graph G, int* D) { // Find smallest value
    int v; // Initialize v to any unvisited vertex
    if (G.getMark(i) == UNVISITED)
        { v = i; break; }
    for (int i=0; i<G.n(); i++) // Find smallest value
    { if ((G.getMark(i) == UNVISITED) && (D[i] < D[v]))
            v = i;
        return v;
    }
```

This is an example of a **greedy** algorithm.
Alternative Prim’s Implementation (1)

Like Dijkstra’s algorithm, can implement with priority queue.

```c
void Prim(Graph G, int s) {
    int v; // The current vertex
    int D[G.n()]; // Distance array
    int V[G.n()]; // Who’s closest
    Elem temp;
    Elem E[G.e()]; // Heap array
    temp.distance = 0; temp.vertex = s;
    E[0] = temp; // Initialize heap array
    heap H(E, 1, G.e()); // Create the heap
    for (int i=0; i<G.n(); i++) D[i] = INFINITY;
    D[s] = 0;
    for (i=0; i<G.n(); i++) { // Now build MST
        temp = H.removemin(); v = temp.vertex;
        while (G.getMark(v) == VISITED);
        G.setMark(v, VISITED);
        if (v != s) AddEdgetoMST(V[v], v);
        if (D[v] == INFINITY) return; // Unreachable
        for (Edge w = each neighbor of v)
            if (D[G.v2(w)] > G.weight(w)) { // Update D
                D[G.v2(w)] = G.weight(w);
                V[G.v2(w)] = v; // Who came from
                temp.distance = D[G.v2(w)];
                temp.vertex = G.v2(w);
                H.insert(temp); // Insert dist in heap
            }
    }
}
```

Kruskal’s MST Algorithm (1)

```c
Kruskal(Graph G) { // Kruskal’s MST algorithm
    Gentree A(G.n()); // Equivalence class array
    Elem E[G.e()]; // Array of edges for min-heap
    int edgecnt = 0;
    for (int i=0; i<G.n(); i++) // Put edges into E
        for (Edge w = G.first(i); G.isEdge(w); w = G.next(w)) {
            E[edgecnt].weight = G.weight(w);
            E[edgecnt++].edge = w;
        }
    heap H(E, edgecnt, edgecnt); // Heapify edges
    int numMST = G.n(); // Init w/ n equiv classes
    for (i=0; numMST>1; i++) { // Combine
        Elem temp = H.removemin(); // Next cheap edge
        Edge w = temp.edge;
        int v = G.v1(w); int u = G.v2(w);
        if (A.differ(v, u)) { // If different
            A.UNION(v, u); // Combine
            AddEdgetoMST(G.v1(w), G.v2(w)); // Add
            numMST--; // Now one less MST
        }
    }
}
```

How do we compute function \(\text{MSTof}(v)\)?
Solution: UNION-FIND algorithm (Section 4.3).
Kruskal’s Algorithm Example

Total cost: \( \Theta(|V| + |E| \log |E|) \).

Proof

Cost is dominated by the edge sort.
Alternative: Use a min heap, quit when only one set left.
“K’th-smallest” implementation.

Matching

- Suppose there are \( n \) workers that we want to work in teams of two. Only certain pairs of workers are willing to work together.
- Problem: Form as many compatible non-overlapping teams as possible.
- Model using \( G \), an undirected graph.
  - A matching is a set of edges in \( G \) with no vertex in more than one edge (the edges are independent).
  - A maximal matching has no free pairs of vertices that can extend the matching.
  - A maximum matching has the greatest possible number of edges.
  - A perfect matching includes every vertex.

Very Dense Graphs (1)

Theorem: Let \( G = (V, E) \) be an undirected graph with \( |V| = 2n \) and every vertex having degree \( \geq n \). Then \( G \) contains a perfect matching.

Proof: Suppose that \( G \) does not contain a perfect matching.
- Let \( M \subseteq E \) be a max matching. \( |M| < n \).
- There must be two unmatched vertices \( v_1, v_2 \) that are not adjacent.
- Every vertex adjacent to \( v_1 \) or to \( v_2 \) is matched.
- Let \( M' \subseteq M \) be the set of edges involved in matching the neighbors of \( v_1 \) and \( v_2 \).
- There are \( \geq 2n \) edges from \( v_1 \) and \( v_2 \) to vertices covered by \( M' \), but \( |M'| < n \).

Very Dense Graphs (2)

Proof: (continued)
- Thus, some edge of \( M' \) is adjacent to 3 edges from \( v_1 \) and \( v_2 \).
- Let \((u_1, u_2)\) be such an edge.
- Replacing \((u_1, u_2)\) with \((v_1, u_2)\) and \((v_2, u_1)\) results in a larger matching.
- Theorem proven by contradiction.

Pigeonhole Principle
Generalizing the Insight

- $v_1, u_2, u_1, v_2$ is a path from an unmatched vertex to an unmatched vertex such that alternate edges are unmatched and matched.
- In one step, switch unmatched and matched edges.
- Let $G = (V, E)$ be an undirected graph and $M \subseteq E$ a matching.
- An alternating path $P$ goes from $v$ to $u$, consists of alternately matched and unmatched edges, and both $v$ and $u$ are not in the match.

Proof: (continued)

- Since $G'$ is a subgraph of $G$ having maximum degree $\leq 2$.

The Alternating Path Theorem (1)

**Theorem**: A matching is maximum iff it has no alternating paths.

**Proof**:
- Clearly, if a matching has alternating paths, then it is not maximum.
- Suppose $M$ is a non-maximum matching.
- Let $M'$ be any maximum matching. Then, $|M'| > |M|$.
- Let $M \oplus M'$ be the symmetric difference of $M$ and $M'$.

$$M \oplus M' = M \cup M' - (M \cap M').$$

- $G' = (V, M \oplus M')$ is a subgraph of $G$ having maximum degree $\leq 2$.

The Alternating Path Theorem (2)

**Proof**: (continued)
- Therefore, the connected components of $G'$ are either even-length cycles or a path with alternating edges.
- Since $|M'| > |M|$, there must be a component of $G'$ that is an alternating path having more $M'$ edges than $M$ edges.

1, 2, 3, 5 is NOT an alternating path (it does not start with an unmatch vertex).

7, 6, 11, 10, 9, 8 is an alternating path with respect to the given matching.

Observation: If a matching has an alternating path, then the size of the matching can be increased by one by switching matched and unmatched edges along the alternating path.

The first point is the obvious part of the iff. If there is an alternating path, simply switch the match and unmatch edges to augment the match.

Symmetric difference: Those in either, but not both.

A vertex matches one different vertex in $M$ and $M'$.
Bipartite Matching

- A bipartite graph $G = (U, V, E)$ consists of two disjoint sets of vertices $U$ and $V$ together with edges $E$ such that every edge has an endpoint in $U$ and an endpoint in $V$.
- Bipartite matching naturally models a number of assignment problems, such as assignment of workers to jobs.
- Alternating paths will work to find a maximum bipartite matching. An alternating path always has one end in $U$ and the other in $V$.
- If we direct unmatched edges from $U$ to $V$ and matched edges from $V$ to $U$, then a directed path from an unmatched vertex in $U$ to an unmatched vertex in $V$ is an alternating path.

Construct BFS subgraph from the set of unmatched vertices

\[ \text{disjoint} \]

1, 6, 3, 7, 4, 9 and 2, 8, 5, 10 are distinguished source vertex sets.

Every edge $(u, v)$ is a directed edge from $u$ to $v$.

Greedily select a maximal set of disjoint alternating paths.

Time complexity $O((|U| + |E|)\sqrt{|V|})$.

Algorithm for Maximum Bipartite Matching

Order doesn't matter. Find a path, remove its vertices, then repeat. Augment along the paths independently since they are disjoint.

Network Flows

Models distribution of utilities in networks such as oil pipelines, waters systems, etc. Also, highway traffic flow.

Simplest version:

A network is a directed graph $G = (V, E)$ having a distinguished source vertex $s$ and a distinguished sink vertex $t$. Every edge $(u, v)$ of $G$ has a capacity $c(u, v) \geq 0$. If $(u, v) \notin E$, then $c(u, v) = 0$. 
Network Flow Definitions

A flow in a network is a function \( f : V \times V \rightarrow R \) with the following properties.

(i) **Skew Symmetry:**
\[ \forall v, w \in V, \quad f(v, w) = -f(w, v). \]

(ii) **Capacity Constraint:**
\[ \forall v, w \in V, \quad f(v, w) \leq c(v, w). \]
If \( f(v, w) = c(v, w) \) then \( (v, w) \) is saturated.

(iii) **Flow Conservation:**
\[ \forall v \in V - \{s, t\}, \quad \sum_{u} f(u, v) = \sum_{w} f(v, w). \]
In other words, flow into \( v \) equals flow out of \( v \).

Flow Example

Edges are labeled “capacity, flow.”
Can omit edges w/o capacity and non-negative flow.
The value of a flow is
\[ |f| = \sum_{w \in V} f(s, w) = \sum_{w \in V} f(w, t). \]

Max Flow Problem

**Problem:** Find a flow of maximum value.

Cut \((X, X')\) is a partition of \( V \) such that \( s \in X, t \in X' \).

The capacity of a cut is
\[ c(X, X') = \sum_{v \in X, w \in X'} c(v, w). \]

A min cut is a cut of minimum capacity.
Cut Flows

For any flow \( f \), the flow across a cut is:
\[
f(X, X') = \sum_{v \in X, w \in X'} f(v, w).
\]

Lemma: For all flows \( f \) and all cuts \( (X, X') \), \( f(X, X') = |f| \).
- Clearly, the flow out of \( s = |f| \) = the flow into \( t \).
- It can be proved that the flow across every other cut is also \( |f| \).

Corollary: The value of any flow is less than or equal to the capacity of a min cut.

Residual Graph

Given any flow \( f \), the residual capacity of the edge is
\[
\text{res}(v, w) = c(v, w) - f(v, w) \geq 0.
\]

Residual graph is a network \( R = (V, E_R) \) where \( E_R \) contains edges of non-zero residual capacity.

Observations

- Any flow in \( R \) can be added to \( F \) to obtain a larger flow in \( G \).
- In fact, a max flow \( f' \) in \( R \) plus the flow \( f \) (written \( f + f' \)) is a max flow in \( G \).
- Any path from \( s \) to \( t \) in \( R \) can carry a flow equal to the smallest capacity of any edge on it.
  - Such a path is called an augmenting path.
  - For example, the path \( s, 1, 2, t \) can carry a flow of 2 units = \( c(1, 2) \).

Max-flow Min-cut Theorem

The following are equivalent:

(i) \( f \) is a max flow.
(ii) \( f \) has no augmenting path in \( R \).
(iii) \( |f| = c(X, X') \) for some min cut \( (X, X') \).

Proof:

(i) \( \Rightarrow \) (ii): If \( f \) has an augmenting path, then \( f \) is not a max flow.
Max-flow Min-cut Theorem (2)

(ii) $\Rightarrow$ (iii):
- Suppose $f$ has no augmenting path in $R$.
- Let $X$ be the subset of $V$ reachable from $s$ and $X' = V - X$.
- Then $s \in X$, $t \in X'$, so $(X, X')$ is a cut.
- $\forall v \in X, w \in X'$, $res(v, w) = c(v, w) - f(v, w) = 0$.
- $f(X, X') = \sum_{v \in X, w \in X'} f(v, w) = \sum_{v \in X, w \in X} c(v, w) = c(X, X')$.
- By Lemma, $|f| = c(X, X')$ and $(X, X')$ is a min cut.

Max-flow Min-cut Theorem (3)

(iii) $\Rightarrow$ (i)
- Let $f$ be a flow such that $|f| = c(X, X')$ for some (min) cut $(X, X')$.
- By Lemma, all flows $f'$ satisfy $|f'| \leq c(X, X') = |f|$.

Thus, $f$ is a max flow.

Max-flow Min-cut Corollary

Corollary: The value of a max flow equals the capacity of a min cut.
This suggests a strategy for finding a max flow.

$R = G; f = 0$;
repeat:
find a path from $s$ to $t$ in $R$;
augment along path to get a larger flow $f$;
update $R$ for new flow;
until $R$ has no path $s$ to $t$.

This is the Ford-Fulkerson algorithm.

If capacities are all rational, then it always terminates with $f$ equal to max flow.

Edmonds-Karp Algorithm

For integral capacities.
Select an augmenting path in $R$ with minimum number of edges.

Performance: $O(|V|^3)$.

There are numerous other approaches to finding augmenting paths, giving a variety of different algorithms.

Network flow remains an active research area.