CS 5114: Theory of Algorithms

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Review of Mathematical Induction

- The paradigm of Mathematical Induction can be used to solve an enormous range of problems.
- Purpose: To prove a parameterized theorem of the form:
  Theorem: \( \forall n \geq c. \ P(n) \).
  Use only positive integers \( \geq c \) for \( n \).
- Sample \( P(n) \):
  \( n + 1 \leq n^2 \)

Principle of Mathematical Induction

- IF the following two statements are true:
  1. \( P(c) \) is true.
  2. For \( n > c \), \( P(n-1) \) is true \( \rightarrow \) \( P(n) \) is true.
  ... THEN we may conclude: \( \forall n \geq c. \ P(n) \).
- The assumption “\( P(n-1) \) is true” is the induction hypothesis.
- Typical induction proof form:
  1. Base case
  2. State induction Hypothesis
  3. Prove the implication (induction step)
- What does this remind you of?

Creation of algorithms comes through exploration, discovery, techniques, intuition: largely by lots of examples and lots of practice (HW exercises). We will use Analysis of Algorithms as a tool. Problem statement (in the software eng. sense) is not important because our problems are easily described, if not easily solved. Smaller problems may or may not be the same as the original problem. Divide and conquer is a way of solving a problem by solving one more more smaller problems. Claim on induction: The processes of constructing proofs and constructing algorithms are similar.

\( P(n) \) is a statement containing \( n \) as a variable.

This sample \( P(n) \) is true for \( n \geq 2 \), but false for \( n = 1 \).

Important: The goal is to prove the implication, not the theorem! That is, prove that \( P(n-1) \rightarrow P(n) \). NOT to prove \( P(n) \). This is much easier, because we can assume that \( P(n) \) is true.

Consider the truth table for implication to see this. Since \( A \rightarrow B \) is (vacuously) true when \( A \) is false, we can just assume that \( A \) is true since the implication is true anyway \( A \) is false. That is, we only need to worry that the implication could be false if \( A \) is true.

The power of induction is that the induction hypothesis “comes for free.” We often try to make the most of the extra information provided by the induction hypothesis. This is like recursion! There you have a base case and a recursive call that must make progress toward the base case.
Induction Example 1

**Theorem:** Let

\[ S(n) = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n. \]

Then, \( \forall n \geq 1, S(n) = \frac{n(n+1)}{2}. \)

Induction Example 2

**Theorem:** \( \forall n \geq 1, \forall x \text{ real} \) such that \( 1 + x > 0, \)

\( (1 + x)^n \geq 1 + nx. \)

Induction Example 3

**Theorem:** 2c and 5c stamps can be used to form any denomination (for denominations \( \geq 4 \)).

Colorings

4-color problem: For any set of polygons, 4 colors are sufficient to guarantee that no two adjacent polygons share the same color.

Restrict the problem to regions formed by placing (infinite) lines in the plane. How many colors do we need?

Candidates:
- 4: Certainly
- 3: ?
- 2: ?
- 1: No!

Let’s try it for 2...
Two-coloring Problem

Given: Regions formed by a collection of (infinite) lines in the plane.
Rule: Two regions that share an edge cannot be the same color.

**Theorem:** It is possible to two-color the regions formed by \( n \) lines.

Strong Induction

**IF** the following two statements are true:
- \( \mathbf{P}(c) \)
- \( \mathbf{P}(i), i = 1, 2, \ldots, n - 1 \rightarrow \mathbf{P}(n) \),

\[ \text{... \ THEN we may conclude: } \forall n \geq c, \mathbf{P}(n). \]

**Advantage:** We can use statements other than \( \mathbf{P}(n - 1) \) in proving \( \mathbf{P}(n) \).

Graph Problem

An **Independent Set** of vertices is one for which no two vertices are adjacent.

**Theorem:** Let \( G = (V, E) \) be a directed graph. Then, \( G \) contains some independent set \( S(G) \) such that every vertex can be reached from a vertex in \( S(G) \) by a path of length at most 2.

Example: a graph with 3 vertices in a cycle. Pick any one vertex as \( S(G) \).

Graph Problem (cont)

**Theorem:** Let \( G = (V, E) \) be a directed graph. Then, \( G \) contains some independent set \( S(G) \) such that every vertex can be reached from a vertex in \( S(G) \) by a path of length at most 2.

**Base Case:** Easy if \( n \leq 3 \) because there can be no path of length > 2.

**Induction Hypothesis:** The theorem is true if \( |V| < n \).

**Induction Step** (\( n > 3 \)):

Pick any \( v \in V \).

Define: \( N(v) = \{v\} \cup \{w \in V | (v, w) \in E\} \).

\( H = G - N(v) \).

Since the number of vertices in \( H \) is less than \( n \), there is an independent set \( S(H) \) that satisfies the theorem for \( H \).
There are two cases:

1. \( S(H) \cup \{v\} \) is independent.
   Then \( S(G) = S(H) \cup \{v\} \).

2. \( S(H) \cup \{v\} \) is not independent.
   Let \( w \in S(H) \) such that \((w,v) \in E\).
   Every vertex in \( N(v) \) can be reached by \( w \) with path of length \( \leq 2 \).
   So, set \( S(G) = S(H) \).

By Strong Induction, the theorem holds for all \( G \).

**Fibonacci Numbers**

Define Fibonacci numbers inductively as:

\[
F(1) = F(2) = 1
\]
\[
F(n) = F(n-1) + F(n-2), \quad n > 2.
\]

**Theorem:** \( \forall n \geq 1, F(n)^2 + F(n+1)^2 = F(2n+1) \).

Induction Hypothesis:
\[
F(n-1)^2 + F(n)^2 = F(2n-1).
\]

With a stronger theorem comes a stronger IH!

**Theorem:**
\[
F(n)^2 + F(n+1)^2 = F(2n+1) \quad \text{and} \quad F(n)^2 + 2F(n)F(n-1) = F(2n).
\]

Induction Hypothesis:
\[
F(n-1)^2 + F(n)^2 = F(2n-1) \quad \text{and} \quad F(n-1)^2 + 2F(n-1)F(n-2) = F(2n-2).
\]

**Another Example**

**Theorem:** All horses are the same color.

**Proof:** \( P(n) \):
If \( S \) is a set of \( n \) horses, then all horses in \( S \) have the same color.

**Base case:** \( n = 1 \) is easy.

**Induction Hypothesis:** Assume \( P(i), i < n \).

**Induction Step:**
- Let \( S \) be a set of horses, \(|S| = n\).
- Let \( S' = S - \{h\} \) for some horse \( h \).
- By IH, all horses in \( S' \) have the same color.
- Let \( h' \) be some horse in \( S' \).
- IH implies \( \{h, h'\} \) have all the same color.

Therefore, \( P(n) \) holds.

"\( S(H) \cup \{v\} \) is not independent" means that there is an edge from something in \( S(H) \) to \( v \).

**IMPORTANT:** There cannot be an edge from \( v \) to \( S(H) \) because whatever we can reach from \( v \) is in \( N(v) \) and would have been removed in \( H \).

We need strong induction for this proof because we don’t know how many vertices are in \( N(v) \).

Expand both sides of the theorem, then cancel like terms:

\[
F(n)^2 + F(n+1)^2 = F(2n+1) \quad \text{and} \quad F(n)^2 + 2F(n)F(n-1) = F(2n) + 2F(n)F(n-1) \]

Want: \( F(n)^2 + F(n+1)^2 = F(2n+1) = F(2n) + 2F(n)F(n-1) \)
Steps above gave:
\[
F(2n) = F(2n-1) + F(2n-2) = 2F(n) + F(n-1) + F(n-1) + F(n-2) \]

So we need to show that:
\[
F(n)^2 + 2F(n)F(n-1) = F(2n) \]

To prove the original theorem, we must prove this. Since we must do it anyway, we should take advantage of this in our IH!

\[
F(n)^2 + 2F(n)F(n-1) = F(2n) \]

... which proves the theorem. The original result could not have been proved without the stronger induction hypothesis.
Algorithm Analysis

- We want to “measure” algorithms.
- What do we measure?

- What factors affect measurement?

- Objective: Measures that are independent of all factors except input.

Time Complexity

- Time and space are the most important computer resources.
- Function of input: \( T(\text{input}) \)
- Growth of time with size of input:
  - Establish an (integer) \( \text{size } n \) for inputs
  - \( n \) numbers in a list
  - \( n \) edges in a graph
- Consider time for all inputs of size \( n \):
  - Time varies widely with specific input
  - Best case
  - Average case
  - Worst case
- Time complexity \( T(n) \) counts steps in an algorithm.

Asymptotic Analysis

- It is undesirable/impossible to count the exact number of steps in most algorithms.
  - Instead, concentrate on main characteristics.
- Solution: Asymptotic analysis
  - Ignore small cases:
    - Consider behavior approaching infinity
  - Ignore constant factors, low order terms:
    - \( 2n^2 \) looks the same as \( 5n^2 + n \) to us.

O Notation

O notation is a measure for “upper bound” of a growth rate.
- pronounced “Big-oh”

Definition: For \( T(n) \) a non-negatively valued function, \( T(n) \) is in the set \( O(f(n)) \) if there exist two positive constants \( c \) and \( n_0 \) such that \( T(n) \leq cf(n) \) for all \( n > n_0 \).

Examples:
- \( 5n + 8 \in O(n) \)
- \( 2n^2 + n \log n \in O(n^2) \in O(n^3 + 5n^2) \)
- \( 2n^2 + n \log n \in O(n^2) \in O(n^3 + n^2) \)

What do we measure?
Time and space to run; ease of implementation (this changes with language and tools); code size

What affects measurement?
Computer speed and architecture; Programming language and compiler; System load; Programmer skill; Specifics of input (size, arrangement)

If you compare two programs running on the same computer under the same conditions, all the other factors (should) cancel out. Want to measure the relative efficiency of two algorithms without needing to implement them on a real computer.

Sometimes analyze in terms of more than one variable. Best case usually not of interest.
Average case usually what we want, but can be hard to measure.
Worst case appropriate for “real-time” applications, often best we can do in terms of measurement.
Examples of “steps:” comparisons, assignments, arithmetic/logical operations. What we choose for “step” depends on the algorithm. Step cost must be “constant” – not dependent on \( n \).

Undesirable to count number of machine instructions or steps because issues like processor speed muddy the waters.

Remember: The time equation is for some particular set of inputs – best, worst, or average case.
O Notation (cont)

We seek the “simplest” and “strongest” $f$.

Big-$O$ is somewhat like $\leq$: $n^2 \in O(n^2)$ and $n^2 \log n \in O(n^2)$, but

- $n^2 \neq n^2 \log n$
- $n^2 \in O(n^2)$ while $n^2 \log n \notin O(n^2)$

A common misunderstanding:

- “The best case for my algorithm is $n = 1$ because that is the fastest.” \textbf{WRONG!}
- Big-$oh$ refers to a growth rate as $n$ grows to $\infty$.
- Best case is defined for the input of size $n$ that is cheapest among all inputs of size $n$.

2$^n$ is an exponential algorithm. 10$n$ and 20$n$ differ only by a constant.


For $n^2$, if $n = 1000$, then $n'$ would be 1003.

Compare $T(n) = n^2$ to $T(n) = n \log n$. For $n > 58$, is faster to have the $O(n \log n)$ algorithm than to have a computer that is 10 times faster.

2$^n$ is an exponential algorithm. 10$n$ and 20$n$ differ only by a constant.

Some Rules for Use

Definition: $f$ is \textbf{monotonically growing} if $n_1 \geq n_2$ implies $f(n_1) \geq f(n_2)$.

We typically assume our time complexity function is monotonically growing.

Theorem 3.1: Suppose $f$ is monotonically growing.

$\forall c > 0$ and $\forall a > 1, (f(n))^c \in O(a^{f(n)})$.

In other words, an \textbf{exponential} function grows faster than a \textbf{polynomial} function.

Lemma 3.2: If $f(n) \in O(s(n))$ and $g(n) \in O(r(n))$ then

- $f(n) + g(n) \in O(s(n) + r(n)) \equiv O(\max(s(n), r(n)))$
- $f(n)g(n) \in O(s(n)r(n))$
- $\text{If } s(n) \in O(h(n)) \text{ then } f(n) \in O(h(n))$
- For any constant $k, f(n) \in O(ks(n))$

Assume monotonically growing because larger problems should take longer to solve. However, many real problems have “cyclically growing” behavior.

Is $O(2^{f(n)}) \in O(3^{f(n)})$? Yes, but not vice versa.

$3^n = 1.5^n \times 2^n$ so no constant could ever make $2^n$ bigger than $3^n$ for all $n$. functional composition
Other Asymptotic Notation

\( \Omega(f(n)) \) – lower bound (\( \geq \))

**Definition:** For \( T(n) \) a non-negatively valued function, \( T(n) \) is in the set \( \Omega(g(n)) \) if there exist two positive constants \( c \) and \( n_0 \) such that \( T(n) \geq cg(n) \) for all \( n > n_0 \).

Ex: \( n^2 \log n \in \Omega(n^2) \).

\( \Theta(f(n)) \) – Exact bound (\( = \))

**Definition:** \( g(n) = \Theta(f(n)) \) if \( g(n) \in O(f(n)) \) and \( g(n) \in \Omega(f(n)) \).

**Important:** It is \( \Theta \) if it is both in \( O \) and in \( \Omega \).

Ex: \( 5n^3 + 4n^2 + 9n + 7 = \Theta(n^3) \)

\( o(f(n)) \) – little o (\(<\))

**Definition:** \( g(n) \in o(f(n)) \) if \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0 \)

Ex: \( n^2 \in o(n^3) \)

\( \omega(f(n)) \) – little omega (\( > \))

**Definition:** \( g(n) \in \omega(f(n)) \) if \( f(n) \in o(g(n)) \).

Ex: \( n^5 \in \omega(n^3) \)

\( \infty(f(n)) \)

**Definition:** \( T(n) = \infty(f(n)) \) if \( T(n) = \Omega(f(n)) \) but the constant in \( \Omega \) is so large that the algorithm is impractical.

\( \Omega \) is most useful to discuss cost of problems, not algorithms. Once you have an equation, the bounds have met. So this is more interesting when discussing your level of uncertainty about the difference between the upper and lower bound.

You have \( \Theta \) when you have the upper and the lower bounds meeting. So \( \Theta \) means that you know a lot more than just Big-oh, and so is preferred when possible.

A common misunderstanding:

- Confusing worst case with upper bound.
- Upper bound refers to a growth rate.
- Worst case refers to the worst input from among the choices for possible inputs of a given size.

We won’t use these too much.

Other Asymptotic Notation (cont)

\( \Omega(f(n)) \) – lower bound (\( \geq \))

**Definition:** For \( T(n) \) a non-negatively valued function, \( T(n) \) is in the set \( \Omega(g(n)) \) if there exist two positive constants \( c \) and \( n_0 \) such that \( T(n) \geq cg(n) \) for all \( n > n_0 \).

Ex: \( n^2 \log n \in \Omega(n^2) \).

\( \Theta(f(n)) \) – Exact bound (\( = \))

**Definition:** \( g(n) = \Theta(f(n)) \) if \( g(n) \in O(f(n)) \) and \( g(n) \in \Omega(f(n)) \).

**Important:** It is \( \Theta \) if it is both in \( O \) and in \( \Omega \).

Ex: \( 5n^3 + 4n^2 + 9n + 7 = \Theta(n^3) \)

\( o(f(n)) \) – little o (\(<\))

**Definition:** \( g(n) \in o(f(n)) \) if \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0 \)

Ex: \( n^2 \in o(n^3) \)

\( \omega(f(n)) \) – little omega (\( > \))

**Definition:** \( g(n) \in \omega(f(n)) \) if \( f(n) \in o(g(n)) \).

Ex: \( n^5 \in \omega(n^3) \)

\( \infty(f(n)) \)

**Definition:** \( T(n) = \infty(f(n)) \) if \( T(n) = \Omega(f(n)) \) but the constant in \( \Omega \) is so large that the algorithm is impractical.

**Aim of Algorithm Analysis**

Typically want to find “simple” \( f(n) \) such that \( T(n) = \Theta(f(n)) \).

- Sometimes we settle for \( O(f(n)) \).

Usually we measure \( T \) as “worst case” time complexity. Sometimes we measure “average case” time complexity.

**Approach:** Estimate number of “steps”

- Appropriate step depends on the problem.
- Ex: measure key comparisons for sorting

**Summation:** Since we typically count steps in different parts of an algorithm and sum the counts, techniques for computing sums are important (loops).

**Recurrence Relations:** Used for counting steps in recursion.

Summation: Guess and Test

**Technique 1:** Guess the solution and use induction to test.

**Technique 1a:** Guess the form of the solution, and use simultaneous equations to generate constants. Finally, use induction to test.
Summation Example

\[ S(n) = \sum_{i=0}^{n} i^2. \]

Guess that \( S(n) \) is a polynomial \( \leq n^3 \).
Equivalently, guess that it has the form
\[ S(n) = an^3 + bn^2 + cn + d. \]

For \( n = 0 \) we have \( S(n) = 0 \) so \( d = 0 \).
For \( n = 1 \) we have \( a + b + c = 1 \).
For \( n = 2 \) we have \( 8a + 4b + 2c = 5 \).
For \( n = 3 \) we have \( 27a + 9b + 3c = 14 \).

Solving these equations yields \( a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6} \).

Now, prove the solution with induction.

Technique 2: Shifted Sums

Given a sum of many terms, shift and subtract to eliminate intermediate terms.
\[ G(n) = \sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n \]
Shift by multiplying by \( r \).
\[ rG(n) = ar + ar^2 + \cdots + ar^n + ar^{n+1} \]
Subtract.
\[ G(n) - rG(n) = G(n)(1 - r) = a - ar^{n+1} \]
\[ G(n) = \frac{a - ar^{n+1}}{1 - r}, \quad r \neq 1 \]

Example 3.3

\[ G(n) = \sum_{i=1}^{n} i2^i = 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n \]
Multiply by 2.
\[ 2G(n) = 1 \times 2^2 + 2 \times 2^3 + 3 \times 2^4 + \cdots + n \times 2^{n+1} \]
Subtract (Note: \( \sum_{i=1}^{n} i2^i = 2^{n+1} - 2 \))
\[ 2G(n) - G(n) = n2^{n+1} - 2^n \cdots 2^2 - 2 \]
\[ G(n) = \frac{n2^{n+1} - 2^n \cdots 2^2 + 2}{1} \]
\[ = (n - 1)2^{n+1} + 2 \]

Recurrence Relations

- A (math) function defined in terms of itself.
- Example: Fibonacci numbers:
  \[ F(n) = F(n-1) + F(n-2) \] (general case)
  \[ F(1) = F(2) = 1 \] (base cases)
- There are always one or more general cases and one or more base cases.
- We will use recurrences for time complexity of recursive (computer) functions.
- General format is \( T(n) = E(T, n) \) where \( E(T, n) \) is an expression in \( T \) and \( n \).
  - \( T(n) = 2T(n/2) + n \)
  - Alternately, an upper bound: \( T(n) \leq E(T, n) \).

This is Manber Problem 2.5.

We need to prove by induction since we don’t know that the guessed form is correct. All that we know without doing the proof is that the form we guessed models some low-order points on the equation properly.

We often solve summations in this way – by multiplying by something or subtracting something. The big problem is that it can be a bit like finding a needle in a haystack to decide what “move” to make. We need to do something that gives us a new sum that allows us either to cancel all but a constant number of terms, or else converts all the terms into something that forms an easier summation.

Shift by multiplying by \( r \) is a reasonable guess in this example since the terms differ by a factor of \( r \). 

We won’t spend a lot of time on techniques... just enough to be able to use them.
Solving Recurrences

We would like to find a closed form solution for $T(n)$ such that:

$$T(n) = \Theta(f(n))$$

Alternatively, find lower bound

- Not possible for inequalities of form $T(n) \leq E(T, n)$.

Methods:
- Guess (and test) a solution
- Expand recurrence
- Theorems

**Guessing**

$$T(n) = 2T(n/2) + 5n^2 \quad n \geq 2$$
$$T(1) = 7$$

Note that $T$ is defined only for powers of 2.

Guess a solution: $T(n) \leq c_1 n^3 = f(n)$

$T(1) = 7$ implies that $c_1 \geq 7$

Inductively, assume $T(n/2) \leq f(n/2)$.

$$T(n) = 2T(n/2) + 5n^2$$
$$\leq 2c_1(n/2)^3 + 5n^2$$
$$\leq c_1(n^3/4) + 5n^2$$
$$\leq c_1 n^3$$ if $c_1 \geq 20/3$.

**Guessing (cont)**

Therefore, if $c_1 = 7$, a proof by induction yields:

$$T(n) \leq 7n^3$$
$$T(n) \in O(n^3)$$

Is this the best possible solution?

**Guessing (cont)**

Guess again.

$$T(n) \leq c_2 n^2 = g(n)$$

$T(1) = 7$ implies $c_2 \geq 7$.

Inductively, assume $T(n/2) \leq g(n/2)$.

$$T(n) = 2T(n/2) + 5n^2$$
$$\leq 2c_2(n/2)^2 + 5n^2$$
$$= c_2(n^2/2) + 5n^2$$
$$\leq c_2 n^2$$ if $c_2 \geq 10$

Therefore, if $c_2 = 10$, $T(n) \leq 10n^2$. $T(n) = O(n^2)$.

Is this the best possible upper bound?

Note that “finding a closed form” means that we have $f(n)$ that doesn’t include $T$.

Can’t find lower bound for the inequality because you do not know enough... you don’t know how much bigger $E(T, n)$ is than $T(n)$, so the result might not be $\Omega(T(n))$.

Guessing is useful for finding an asymptotic solution. Use induction to prove the guess correct.

For Big-oh, not many choices in what to guess.

$$7 \times 1^3 = 7$$

Because $\frac{20}{3}n^3 + 5n^2 = \frac{20}{3}n^3$ when $n = 1$, and as $n$ grows, the right side grows even faster.

No - try something tighter.

Because $\frac{10}{3}n^2 + 5n^2 = 10n^2$ for $n = 1$, and the right hand side grows faster.

Yes this is best, since $T(n)$ can be as bad as $5n^2$. 
Guessing (cont)

Now, reshape the recurrence so that $T$ is defined for all values of $n$.

$$T(n) \leq 2T(n/2) + 5n^2 \quad n \geq 2$$

For arbitrary $n$, let $2^k - 1 \leq n \leq 2^k$.

We have already shown that $T(2^k) \leq 10(2^k)^2$.

$$T(n) \leq T(2^k) \leq 10(2^k)^2$$

$$= 10(2^k/n)^2n^2 \leq 10(2)^2n^2$$

$$\leq 40n^2$$

Hence, $T(n) = O(n^2)$ for all values of $n$.

Typically, the bound for powers of two generalizes to all $n$.

Expanding Recurrences

Usually, start with equality version of recurrence.

$$T(n) = 2T(n/2) + 5n^2$$

$$T(1) = 7$$

Assume $n$ is a power of 2; $n = 2^k$.

Expanding Recurrences (cont)

$$T(n) = 2T(n/2) + 5n^2$$

$$= 2(2T(n/4) + 5(n/2)^2) + 5n^2$$

$$= 2(2(2T(n/8) + 5(n/4)^2) + 5(n/2)^2) + 5n^2$$

$$= 2^kT(1) + 2^{k-1} \cdot 5(n/2^{k-1})^2 + 2^{k-2} \cdot 5(n/2^{k-2})^2$$

$$\quad + \cdots + 2 \cdot 5(n/2)^2 + 5n^2$$

$$= 7n + 5 \sum_{i=0}^{k-1} n^2/2^i = 7n + 5n^2 \sum_{i=0}^{k-1} 1/2^i$$

$$= 7n + 5n^2(2 - 1/2^{k-1})$$

$$= 7n + 5n^2(2 - 2/n).$$

This it the exact solution for powers of 2. $T(n) = \Theta(n^2)$.

Divide and Conquer Recurrences

These have the form:

$$T(n) = aT(n/b) + cn^k$$

$$T(1) = c$$

... where $a, b, c, k$ are constants.

A problem of size $n$ is divided into $a$ subproblems of size $n/b$, while $cn^k$ is the amount of work needed to combine the solutions.
Divide and Conquer Recurrences (cont)

Expand the sum; $n = b^m$.

$$T(n) = a(n/b)((a/b) + c(n/b) + cr^h) + a^m T(1) + a^{m-1} c(n/b^{m-1}) + \ldots + ac(n/b) + cr^h$$

$$= ca^m \sum_{i=0}^m (b^i/a)^1$$

$$a^m = b^{m+n} = n^{m \log_b a}$$

The summation is a geometric series whose sum depends on the ratio $r = b^k/a$.

There are 3 cases.

D & C Recurrences (cont)

(1) $r < 1$.

$$\sum_{i=0}^m r^i < 1/(1 - r), \quad a \text{ constant.}$$

$$T(n) = \Theta(a^m) = \Theta(n^{m \log_b a}).$$

(2) $r = 1$.

$$\sum_{i=0}^m r^i = m + 1 = \log_b n + 1$$

$$T(n) = \Theta(n^{m \log_b a} \log n) = \Theta(n^k \log n)$$

D & C Recurrences (Case 3)

(3) $r > 1$.

$$\sum_{i=0}^m r^i = \frac{r^{m+1} - 1}{r - 1} = \Theta(r^m)$$

So, from $T(n) = ca^m \sum r^i$,

$$T(n) = \Theta(a^{m^2})$$

$$= \Theta((b^k/a)^m)$$

$$= \Theta((b^k)^m)$$

$$= \Theta(n^k)$$

Summary

Theorem 3.4:

$$T(n) = \begin{cases} 
\Theta(n^{m \log_b a}) & \text{if } a > b^k \\
\Theta(n^k \log n) & \text{if } a = b^k \\
\Theta(n^k) & \text{if } a < b^k 
\end{cases}$$

Apply the theorem:

$$T(n) = 3T(n/5) + 8n^2.$$  
$a = 3, b = 5, c = 8, k = 2.$

$b^k/a = 25/3.$

Case (3) holds: $T(n) = \Theta(n^2)$. We simplify by approximating summations.
Examples

- Mergesort: \( T(n) = 2T(n/2) + n \).
  
- Binary search: \( T(n) = T(n/2) + 2 \).

- Insertion sort: \( T(n) = T(n-1) + n \).

- Standard Matrix Multiply (recursively):
  \[ T(n) = 8 T(n/2) + n^2. \]

The cost for \( m \) pushes and \( m \) pops:

\[ m_1 + (m_2 + m_1) = O(m_1 + m_2) \]

Future potential is not available for future pop operations.

Useful log Notation

- If you want to take the log of \((\log n)\), it is written \(\log\log n\).
- \((\log n)^2\) can be written \(\log^2 n\).

Amortized Analysis

Consider this variation on STACK:

```c
void init(STACK S);
element examineTop(STACK S);
void push(element x, STACK S);
void pop(int k, STACK S);
...
```

... where pop removes \( k \) entries from the stack.

“Local” worst case analysis for pop:

\( O(n) \) for \( n \) elements on the stack.

Given \( m_1 \) calls to push, \( m_2 \) calls to pop:

Naive worst case: \( m_1 \cdot m_2 \cdot n = m_1 + m_2 \cdot m_1. \)

Alternate Analysis

Use amortized analysis on multiple calls to push, pop:

Cannot pop more elements than get pushed onto the stack.

After many pushes, a single pop has high potential.

Once that potential has been expended, it is not available for future pop operations.

Actual number (constant time) push calls + (Actual number of pop calls + Total potential for the pops)

CLR has an entire chapter on this – we won’t go into this much, but we use Amortized Analysis implicitly sometimes.
Creative Design of Algorithms by Induction

Analogy: Induction ↔ Algorithms

Begin with a problem:

- “Find a solution to problem Q.”

Think of Q as a set containing an infinite number of problem instances.

Example: Sorting

- Q contains all finite sequences of integers.

Solving Q

First step:

- Parameterize problem by size: \( Q(n) \)

Example: Sorting

- \( Q(n) \) contains all sequences of \( n \) integers.

Algorithm: Solve for an instance in \( Q(n) \) by solving instances in \( Q(i), i < n \) and combining as necessary.

Induction

Goal: Prove that we can solve for an instance in \( Q(n) \) by assuming we can solve instances in \( Q(i), i < n \).

Don’t forget the base cases!

Theorem: \( \forall n \geq 1 \), we can solve instances in \( Q(n) \).

- This theorem embodies the correctness of the algorithm.

Since an induction proof is mechanistic, this should lead directly to an algorithm (recursive or iterative).

Just one (new) catch:

- Different inductive proofs are possible.
- We want the most efficient algorithm!

Interval Containment

Start with a list of non-empty intervals with integer endpoints.

Example:

\[ [6, 9], [5, 7], [0, 3], [4, 8], [6, 10], [7, 8], [0, 5], [1, 3], [6, 8] \]

Now that we have completed the tool review, we will do two things:

1. Survey algorithms in application areas
2. Try to understand how to create efficient algorithms

This chapter is about the second. The remaining chapters do the second in the context of the first.

I — A is reasonably obvious — we often use induction to prove that an algorithm is correct. The intellectual claim of Manber is that I — A gives insight into problem solving.
Interval Containment (cont)

Problem: Identify and mark all intervals that are contained in some other interval.

Example:
- Mark $[6, 9]$ since $[6, 9] \subseteq [6, 10]

Interval Containment (cont)

- $Q(n)$: Instances of $n$ intervals
- **Base case**: $Q(1)$ is easy.
- **Inductive Hypothesis**: For $n > 1$, we know how to solve an instance in $Q(n-1)$.
- **Induction step**: Solve for $Q(n)$.
  - Solve for first $n-1$ intervals, applying inductive hypothesis.
  - Check the $n$th interval against intervals $i = 1, 2, \ldots$
  - If interval $i$ contains interval $n$, mark interval $n$. (stop)
  - If interval $n$ contains interval $i$, mark interval $i$.
- **Analysis**:
  \[
  T(n) = T(n-1) + cn \\
  T(n) = \Theta(n^2)
  \]

“Creative” Algorithm

Idea: Choose a special interval as the $n$th interval.

Choose the $n$th interval to have rightmost left endpoint, and if there are ties, leftmost right endpoint.

(1) No need to check whether $n$th interval contains other intervals.

(2) $n$th interval should be marked iff the rightmost endpoint of the first $n-1$ intervals exceeds or equals the right endpoint of the $n$th interval.

Solution: Sort as above.

“Creative” Solution Induction

**Induction Hypothesis**: Can solve for $Q(n-1)$ AND interval $n$ is the “rightmost” interval AND we know $R$ (the rightmost endpoint encountered so far) for the first $n-1$ segments.

**Induction Step**: (to solve $Q(n)$)
- Solve for first $n-1$ intervals recursively, and remember $R$.
- If the rightmost endpoint of $n$th interval is $\leq R$, then mark the $n$th interval.
- Else $R \leftarrow$ right endpoint of $n$th interval.

**Analysis**: $\Theta(n \log n) + \Theta(n)$.

**Lesson**: Preprocessing, often sorting, can help sometimes.
Maximal Induced Subgraph

**Problem:** Given a graph $G = (V, E)$ and an integer $k$, find a maximal induced subgraph $H = (U, F)$ such that all vertices in $H$ have degree $\geq k$.

Example: Scientists interacting at a conference. Each one will come only if $k$ colleagues come, and they know in advance if somebody won’t come.

Example: For $k = 3$.

Solution:

Max Induced Subgraph Solution

$Q(s, k)$: Instances where $|V| = s$ and $k$ is a fixed integer.

**Theorem:** $\forall s, k > 0$, we can solve an instance in $Q(s, k)$.

**Analysis:** Should be able to implement algorithm in time $\Theta(|V| + |E|)$.

Celebrity Problem

In a group of $n$ people, a **celebrity** is somebody whom everybody knows, but who knows no one else.

**Problem:** If we can ask questions of the form “does person $i$ know person $j$?” how many questions do we need to find a celebrity, if one exists?

How should we structure the information?

Celebrity Problem (cont)

Formulate as an $n \times n$ boolean matrix $M$.

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

Example:

A celebrity has all 0’s in his row and all 1’s in his column.

There can be at most one celebrity.

Clearly, $O(n^2)$ questions suffice. Can we do better?

Celebrity Problem (cont)

The celebrity in this example is 4.
Efficient Celebrity Algorithm

Appeal to induction:
- If we have an \( n \times n \) matrix, how can we reduce it to an \( (n - 1) \times (n - 1) \) matrix?

What are ways to select the \( n \)th person?

Efficient Celebrity Algorithm (cont)

Eliminate one person if he is a non-celebrity.
- Strike one row and one column.

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Does 1 know 3? No. 3 is a non-celebrity.
Does 2 know 5? Yes. 2 is a non-celebrity.
Observation: Each question eliminates one non-celebrity.

Celebrity Algorithm

Algorithm:
- Ask \( n - 1 \) questions to eliminate \( n - 1 \) non-celebrities. This leaves one candidate who might be a celebrity.
- Ask \( 2(n - 1) \) questions to check candidate.

Analysis:
- \( O(n) \) questions are asked.

Example:
- Does 1 know 2? No. Eliminate 2
- Does 1 know 3? No. Eliminate 3
- Does 1 know 4? Yes. Eliminate 1
- Does 4 know 5? No. Eliminate 5

4 remains as candidate.

Maximum Consecutive Subsequence

Given a sequence of integers, find a contiguous subsequence whose sum is maximum.

The sum of an empty subsequence is 0.
- It follows that the maximum subsequence of a sequence of all negative numbers is the empty subsequence.

Example:
\[2, 11, -9, 3, 4, -6, -7, 7, -3, 5, 6, -2\]

Maximum subsequence:
\[7, -3, 5, 6\] Sum: 15
Finding an Algorithm

Induction Hypothesis: We can find the maximum subsequence sum for a sequence of \(< n\) numbers.

Note: We have changed the problem.
- First, figure out how to compute the sum.
- Then, figure out how to get the subsequence that computes that sum.

Finding an Algorithm (cont)

Induction Hypothesis: We can find the maximum subsequence sum for a sequence of \(< n\) numbers.

Let \(S = x_1, x_2, \ldots, x_n\) be the sequence.

Base case: \(n = 1\)
- Either \(x_1 < 0 \Rightarrow \text{sum} = 0\)
- Or \(\text{sum} = x_1\).

Induction Step:
- We know the maximum subsequence \(\text{SUM}(n-1)\) for \(x_1, x_2, \ldots, x_{n-1}\).
- Where does \(x_n\) fit in?
  - Either it is not in the maximum subsequence or it ends the maximum subsequence.
  - If \(x_n\) ends the maximum subsequence, it is appended to trailing maximum subsequence of \(x_1, \ldots, x_{n-1}\).

Finding an Algorithm (cont)

Need: \(\text{TRAILINGSUM}(n-1)\) which is the maximum sum of a subsequence that ends \(x_1, \ldots, x_{n-1}\).

To get this, we need a stronger induction hypothesis.

Maximum Subsequence Solution

New Induction Hypothesis: We can find \(\text{SUM}(n-1)\) and \(\text{TRAILINGSUM}(n-1)\) for any sequence of \(n - 1\) integers.

Base case:
\(\text{SUM}(1) = \text{TRAILINGSUM}(1) = \text{Max}(0, x_1)\).

Induction step:
\(\text{SUM}(n) = \text{Max}(\text{SUM}(n-1), \text{TRAILINGSUM}(n-1) + x_n)\).
\(\text{TRAILINGSUM}(n) = \text{Max}(0, \text{TRAILINGSUM}(n-1) + x_n)\).
Maximum Subsequence Solution (cont)

Analysis:
Important Lesson: If we calculate and remember some additional values as we go along, we are often able to obtain a more efficient algorithm.
This corresponds to strengthening the induction hypothesis so that we compute more than the original problem (appears to require).
How do we find sequence as opposed to sum?

The Knapsack Problem

Problem:
- Given an integer capacity $K$ and $n$ items such that item $i$ has an integer size $k_i$, find a subset of the $n$ items whose sizes exactly sum to $K$, if possible.
- That is, find $S \subseteq \{1, 2, \cdots, n\}$ such that $\sum_{i \in S} k_i = K$.

Example:
Knapsack capacity $K = 163$.
10 items with sizes 4, 9, 15, 19, 27, 44, 54, 68, 73, 101

Knapsack Algorithm Approach

Instead of parameterizing the problem just by the number of items $n$, we parameterize by both $n$ and by $K$.

$P(n, K)$ is the problem with $n$ items and capacity $K$.

First consider the decision problem: Is there a subset $S$?

Induction Hypothesis:
We know how to solve $P(n - 1, K)$.

Knapsack Induction

Induction Hypothesis:
We know how to solve $P(n - 1, K)$.

Solving $P(n, K)$:
- If $P(n - 1, K)$ has a solution, then it is also a solution for $P(n, K)$.
- Otherwise, $P(n, K)$ has a solution iff $P(n - 1, K - k_n)$ has a solution.

So what should the induction hypothesis really be?
Knapsack: New Induction

- **New Induction Hypothesis:**
  - We know how to solve \( P(n-1,k), 0 \leq k \leq K \).
- **To solve \( P(n,K) \):**
  - If \( P(n-1,K) \) has a solution,
    - Then \( P(n,K) \) has a solution.
  - Else if \( P(n-1,K-k) \) has a solution,
    - Then \( P(n,K) \) has a solution.
  - Else \( P(n,K) \) has no solution.

Algorithm Complexity

- **Resulting algorithm complexity:**
  
  \[
  T(n) = 2T(n-1) + c \quad n \geq 2 \\
  T(n) = \Theta(2^n) \quad \text{by expanding sum.}
  \]
- **Alternate:** change variable from \( n \) to \( m = 2^n \).
  
  \[2T(m/2) + c_m n^2.\]
  
  From Theorem 3.4, we get \( \Theta(m^{\log_2 5}) = \Theta(2^n) \).
- **But, there are only \( n(K+1) \) problems defined.**
  - It must be that problems are being re-solved many times by this algorithm. Don't do that.

Efficient Algorithm Implementation

The key is to avoid re-computing subproblems.

**Implementation:**

- Store an \( n \times (K+1) \) matrix to contain solutions for all the \( P(i,k) \).
- Fill in the table row by row.
- Alternately, fill in table using logic above.

**Analysis:**

\[ T(n) = \Theta(nK). \]

Space needed is also \( \Theta(nK) \).

Example

\( K = 10 \), with 5 items having size 9, 2, 7, 4, 1.

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 )</td>
<td></td>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_3 )</td>
<td></td>
<td></td>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>( k_4 )</td>
<td></td>
<td></td>
<td></td>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>I/O</td>
</tr>
<tr>
<td>( k_5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>I</td>
</tr>
</tbody>
</table>

Key:

- No solution for \( P(i,k) \)
- \( O \) Solution(s) for \( P(i,k) \) with \( i \) omitted.
- \( I \) Solution(s) for \( P(i,k) \) with \( i \) included.
- \( I/O \) Solutions for \( P(i,k) \) both with \( i \) included and with \( i \) omitted.

Need to solve two subproblems: \( P(n-1,k) \) and \( P(n-1,k-k) \).

Problem: Can't use Theorem 3.4 in this form.
This form uses \( n^2 \) because we also need an exponent of \( n \) to fit the form of the theorem.

To solve \( P(i,k) \) look at entry in the table.
If it is marked, then OK.
Otherwise solve recursively.
Initially, fill in all \( P(i,0) \).

Example: \( M(3, 9) \) contains \( O \) because \( P(2, 9) \) has a solution.
It contains \( I \) because \( P(2, 9) - P(2, 9 - 7) \) has a solution.
How can we find a solution to \( P(5, 10) \) from \( M \)?
How can we find all solutions for \( P(5, 10) \)?
Solution Graph

Find all solutions for $P(5, 10)$.

```
\begin{array}{c}
M(1, 0) & M(1, 9) \\
\uparrow & \uparrow \\
M(2, 2) & M(2, 9) \\
\uparrow & \uparrow \\
M(3, 9) \\
\uparrow \\
M(4, 9) \\
\uparrow \\
M(5, 10)
\end{array}
```

The result is an $n$-level DAG.

Dynamic Programming

This approach of storing solutions to subproblems in a table is called **dynamic programming**.

It is useful when the number of distinct subproblems is not too large, but subproblems are executed repeatedly.

Implementation: Nested for loops with logic to fill in a single entry.

Most useful for **optimization problems**.

Fibonacci Sequence

```c
int Fibr(int n) {
    if (n <= 1) return 1; // Base case
    return Fibr(n-1) + Fibr(n-2); // Recursion
}
```

- Cost is Exponential. Why?
- If we could eliminate redundancy, cost would be greatly reduced.

Fibonacci Sequence (cont)

- Keep a table

```c
int Fibrt(int n, int* Values) {
    // Assume Values has at least n slots, and all slots are initialized to 0
    if (n <= 1) return 1; // Base case
    if (Values[n] == 0) // Compute and store
        Values[n] = Fibrt(n-1, Values) + Fibrt(n-2, Values);
    return Values[n];
}
```

- Cost?
- We don't need table, only last 2 values.
- Key is working bottom up.
**Chained Matrix Multiplication**

**Problem:** Compute the product of \( n \) matrices

\[
M = M_1 \times M_2 \times \cdots \times M_n
\]
as efficiently as possible.

If \( A \) is \( r \times s \) and \( B \) is \( s \times t \), then

\[
\text{COST}(A \times B) =
\text{SIZE}(A \times B) =
\]

If \( C \) is \( t \times u \) then

\[
\text{COST}((A \times B) \times C) =
\text{COST}(A \times (B \times C)) =
\]

**Order Matters**

Example:

\[
A = 2 \times 8; B = 8 \times 5; C = 5 \times 20
\]

\[
\text{COST}((A \times B) \times C) =
\text{COST}(A \times (B \times C)) =
\]

View as binary trees:

**Chained Matrix Induction**

**Induction Hypothesis:** We can find the optimal evaluation tree for the multiplication of \( \leq n - 1 \) matrices.

**Induction Step:** Suppose that we start with the tree for:

\[
M_1 \times M_2 \times \cdots \times M_{n-1}
\]

and try to add \( M_n \).

Two obvious choices:

- Multiply \( M_{n-1} \times M_n \) and replace \( M_{n-1} \) in the tree with a subtree.
- Multiply \( M_n \) by the result of \( P(n-1) \): make a new root.

Visually, adding \( M_n \) may radically order the (optimal) tree.

**Alternate Induction**

**Induction Step:** Pick some multiplication as the root, then recursively process each subtree.

- Which one? Try them all!
- Choose the cheapest one as the answer.
- How many choices?

Observation: If we know the \( i \)th multiplication is the root, then the left subtree is the optimal tree for the first \( i - 1 \) multiplications and the right subtree is the optimal tree for the last \( n - i - 1 \) multiplications.

Notation: for \( 1 \leq i \leq j \leq n \),

\[
c[i, j] = \text{minimum cost to multiply } M_i \times M_{i+1} \times \cdots \times M_j.
\]

So,

\[
c[1, n] = \min_{1 \leq i \leq n-1} n \cdot r_n + c[1, i] + c[i+1, n].
\]
Analysis

Base Cases: For $1 \leq k \leq n$, $c[k, k] = 0$.
More generally:

$$c[i, j] = \min_{1 \leq k < j - 1} n_{k - 1} n_{n_j} + c[i, k] + c[k + 1, j]$$

Solving $c[i, j]$ requires $2(j - i)$ recursive calls.

Analysis:

$$T(n) = \sum_{k=1}^{n-1} (T(k) + T(n-k)) = 2 \sum_{k=1}^{n-1} T(k)$$
$$T(1) = 1$$
$$T(n + 1) = T(n) + 2T(n) = 3T(n)$$
$$T(n) = \Theta (3^n) \; \text{Ugh!}$$

But there are only $\Theta (n^2)$ values $c[i, j]$ to be calculated!

Dynamic Programming

Make an $n \times n$ table with entry $(i, j) = c[i, j]$.


c[1, 1]  c[1, 2]  \cdots  c[1, n] \\
c[2, 2]  \cdots  c[2, n] \\
\vdots  \vdots  \vdots  \vdots  \vdots  \vdots  \vdots \\
c[n, n]

Only upper triangle is used. Fill in table diagonal by diagonal.

$c[i, i] = 0$.

For $1 \leq i < j \leq n$,

$$c[i, j] = \min_{k \leq s \leq j - 1} n_{s-1} n_{n_j} + c[i, k] + c[k + 1, j]$$.

Dynamic Programming Analysis

- The time to calculate $c[i, j]$ is proportional to $j - i$.
- There are $\Theta(n^2)$ entries to fill.
- $T(n) = O(n^3)$.
- Also, $T(n) = \Omega(n^3)$.
- How do we actually find the best evaluation order?

Summary

- Dynamic programming can often be added to an inductive proof to make the resulting algorithm as efficient as possible.
- Can be useful when divide and conquer fails to be efficient.
- Usually applies to optimization problems.
- Requirements for dynamic programming:
  - Small number of subproblems, small amount of information to store for each subproblem.
  - Base case easy to solve.
  - Easy to solve one subproblem given solutions to smaller subproblems.

2 calls for each root choice, with $(j - i)$ choices for root. But, these don’t all have equal cost.

Actually, since $j > i$, only about half that needs to be done.

For middle diagonal of size $n/2$, each costs $n/2$.

For each $c[i, j]$, remember the $k$ (the root of the tree) that minimizes the expression.

So, store in table the next place to go.