CS 5114: Theory of Algorithms

Clifford A. Shaffer
Department of Computer Science
Blacksburg, Virginia

Spring 2010
Copyright © 2010 by Clifford A. Shaffer

Review of Mathematical Induction

- The paradigm of Mathematical Induction can be used to solve an enormous range of problems.
- Purpose: To prove a parameterized theorem of the form:
  \[ \forall n \geq c. \quad P(n). \]
  - Use only positive integers \( \geq c \) for \( n \).
- Sample \( P(n) \):
  \[ n + 1 \leq n^2 \]

Principle of Mathematical Induction

- IF the following two statements are true:
  1. \( P(c) \) is true.
  2. For \( n > c. \quad P(n - 1) \) is true \( \rightarrow P(n) \) is true.
  THEN we may conclude: \( \forall n \geq c. \quad P(n) \).
- The assumption “\( P(n - 1) \) is true” is the induction hypothesis.
- Typical induction proof form:
  1. Base case
  2. State induction hypothesis
  3. Prove the implication (induction step)
- What does this remind you of?
Induction Example 1

Theorem: Let

\[ S(n) = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n. \]

Then, \( \forall n \geq 1, S(n) = \frac{n(n+1)}{2}. \)

Induction Example 2

Theorem: \( \forall n \geq 1, \forall x \text{ real such that } 1 + x > 0, \]

\[ (1 + x)^n \geq 1 + nx. \]

Base Case: \( (1 + x)^1 = 1 + x \geq 1 + x \\
Induction Hypothesis: Assume \( (1 + x)^{n-1} \geq 1 + (n - 1)x \)

Induction Step:

\[ (1 + x)^n = (1 + x)(1 + x)^{n-1} \geq (1 + x)(1 + (n - 1)x) = 1 + nx - x + nx^2 - x^2 = 1 + nx + (n - 1)x^2 \geq 1 + nx. \]

Induction Example 3

Theorem: 2c and 5c stamps can be used to form any denomination (for denominations \( \geq 4 \)).

Base case: \( 4 = 2 + 2. \)

Induction Hypothesis: Assume \( P(k) \) for \( 4 \leq k < n. \)

Induction Step:

Case 1: \( n - 1 \) made up of all 2c stamps. Then, replace 2 of these with a 5c stamp.

Case 2: \( n - 1 \) includes a 5c stamp. Then, replace this with 3 2c stamps.

Colorings

4-color problem: For any set of polygons, 4 colors are sufficient to guarantee that no two adjacent polygons share the same color.

Restrict the problem to regions formed by placing (infinite) lines in the plane. How many colors do we need?

Candidates:

- 4: Certainly
- 3: ?
- 2: ?
- 1: No!

Let’s try it for 2...
Two-coloring Problem

Given: Regions formed by a collection of (infinite) lines in the plane.
Rule: Two regions that share an edge cannot be the same color.

Theorem: It is possible to two-color the regions formed by \( n \) lines.

Strong Induction

IF the following two statements are true:
1. \( P(c) \)
2. \( P(i), i = 1, 2, \ldots, n - 1 \rightarrow P(n) \),

THEN we may conclude: \( \forall n \geq c, P(n) \).

Advantage: We can use statements other than \( P(n - 1) \) in proving \( P(n) \).

Graph Problem

An Independent Set of vertices is one for which no two vertices are adjacent.

Theorem: Let \( G = (V, E) \) be a directed graph. Then, \( G \) contains some independent set \( S(G) \) such that every vertex can be reached from a vertex in \( S(G) \) by a path of length at most 2.

Example: a graph with 3 vertices in a cycle. Pick any one vertex as \( S(G) \).

Graph Problem (cont)

Theorem: Let \( G = (V, E) \) be a directed graph. Then, \( G \) contains some independent set \( S(G) \) such that every vertex can be reached from a vertex in \( S(G) \) by a path of length at most 2.

Base Case: Easy if \( n \leq 3 \) because there can be no path of length \( > 2 \).

Induction Hypothesis: The theorem is true if \( |V| < n \).

Induction Step \((n > 3)\):
Pick any \( v \in V \).
Define: \( N(v) = \{v\} \cup \{w \in V | (v, w) \in E\} \).
\( H = G - N(v) \).

Since the number of vertices in \( H \) is less than \( n \), there is an independent set \( S(H) \) that satisfies the theorem for \( H \).

Picking what to do induction on can be a problem. Lines?
Regions? How can we “add a region?” We can’t, so try induction on lines.

**Base Case:** \( n = 1 \). Any line divides the plane into two regions.

**Induction Hypothesis:** It is possible to two-color the regions formed by \( n - 1 \) lines.

**Induction Step:** Introduce the \( n' \)th line.
This line cuts some colored regions in two.
Reverse the region colors on one side of the \( n' \)th line. A valid two-coloring results.

- Any boundary surviving the addition still has opposite colors.
- Any new boundary also has opposite colors after the switch.

The previous examples were all very straightforward – simply add in the \( n' \)th item and justify that the IH is maintained.
Now we will see examples where we must do more sophisticated (creative!) maneuvers such as
- go backwards from \( n \).
- prove a stronger IH.

to make the most of the IH.

It should be obvious that the theorem is true for an undirected graph.
Naive approach: Assume the theorem is true for any graph of \( n - 1 \) vertices. Now add the \( n \)th vertex and its edges. But this won’t work for the graph \( 1 \rightarrow 2 \). Initially, vertex 1 is the independent set. We can’t add 2 to the graph. Nor can we reach it from 1.

Going forward is good for proving existence.
Going backward (from an arbitrary instance into the IH) is usually necessary to prove that a property holds in all instances. This is because going forward requires proving that you reach all of the possible instances.

\( N(v) \) is all vertices reachable (directly) from \( v \). That is, the Neighbors of \( v \).
\( H \) is the graph induced by \( V - N(v) \).

OK, so why remove both \( v \) and \( N(v) \) from the graph? If we only remove \( v \), we have the same problem as before. If \( G \) is \( 1 \rightarrow 2 \rightarrow 3 \), and we remove 1, then the independent set for \( H \) must be vertex 2. We can’t just add back 1. But if we remove both 1 and 2, then we’ll be able to do something...
Graph Proof (cont)

There are two cases:

1. \( S(H) \cup \{ v \} \) is independent.
   Then \( S(G) = S(H) \cup \{ v \} \).

2. \( S(H) \cup \{ v \} \) is not independent.
   Let \( w \in S(H) \) such that \((w, v) \in E\).
   Every vertex in \( N(v) \) can be reached by \( w \) with path of length \( \leq 2 \).
   So, set \( S(G) = S(H) \).

By Strong Induction, the theorem holds for all \( G \).

Fibonacci Numbers

Define Fibonacci numbers inductively as:

\[
F(1) = F(2) = 1 \\
F(n) = F(n - 1) + F(n - 2), \quad n > 2.
\]

Theorem: \( \forall n \geq 1, \ F(n)^2 + F(n+1)^2 = F(2n+1). \)

Induction Hypothesis:
\( F(n-1)^2 + F(n)^2 = F(2n-1). \)

“\( S(H) \cup \{ v \} \) is not independent” means that there is an edge from something in \( S(H) \) to \( v \).

IMPORTANT: There cannot be an edge from \( v \) to \( S(H) \) because whatever we can reach from \( v \) is in \( N(v) \) and would have been removed in \( H \).

We need strong induction for this proof because we don’t know how many vertices are in \( N(v) \).