Priority Queues

T. M. Murali

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Sort

**INSTANCE:** Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

**SOLUTION:** A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$. 
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▶ Possible algorithm:
  - Store all the numbers in a data structure \( D \).
  - Repeatedly find the smallest number in \( D \), output it, and remove it.
Motivation: Sort a List of Numbers

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Possible algorithm:

- Store all the numbers in a data structure $D$.
- Repeatedly find the smallest number in $D$, output it, and remove it.

To get $O(n \log n)$ running time, each “find minimum” step must take $O(\log n)$ time.
Candidate Data Structures for Sorting

Data structure must support insertion, finding minimum, and deleting minimum.
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**List**  
Insertion and deletion take $O(1)$ time but finding minimum requires scanning the list and takes $\Omega(n)$ time.

**Sorted array**  
Finding minimum takes $O(1)$ time but insertion and deletion can take $\Omega(n)$ time in the worst case.
Priority Queue

▶ Store a set $S$ of elements, where each element $v$ has a priority value $\text{key}(v)$.
▶ Smaller key values $\equiv$ higher priorities.
▶ Operations supported: find the element with smallest key, remove the smallest element, update the key of an element, insert an element, delete an element.
▶ Key update and element deletion require knowledge of the position of the element in the priority queue.
Heaps

- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- **Heap order.** For every element $v$ at a node $i$, the element $w$ at $i$’s parent satisfies $\text{key}(w) \leq \text{key}(v)$.
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- We can implement a heap in a pointer-based data structure.
- Alternatively, assume maximum number \( N \) of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index \( i \) has children at indices \( 2i \) and \( 2i + 1 \) and parent at index \( \lfloor i/2 \rfloor \).
  - Index 1 is the root.
  - How do you know that a node at index \( i \) is a leaf?
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- Alternatively, assume maximum number $N$ of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index $i$ has children at indices $2i$ and $2i + 1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index $i$ is a leaf? If $2i > n$, the number of elements in the heap.
Example of a Heap

Figure 2.3 Values in a heap shown as a binary tree on the left, and represented as an array on the right. The arrows show the children for the top three nodes in the tree.
Inserting an Element: Heapify-up

1. Insert new element at index $n + 1$.
2. Fix heap order using Heapify-up($H, n + 1$).

Heapify-up($H, i$):

If $i > 1$ then

let $j = \text{parent}(i) = \lfloor i/2 \rfloor$

If key[$H[i]$] < key[$H[j]$] then

swap the array entries $H[i]$ and $H[j]$

Heapify-up($H, j$)

Endif

Endif
Example of **Heapify-up**

The **Heapify-up** process is moving element \( v \) toward the root.

**Figure 2.4** The **Heapify-up** process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).
Correctness of Heapify-up

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Correctness of Heapify-up

- **H is almost a heap with key of H[i] too small** if there is a value \( \alpha \geq \text{key}(H[i]) \) such that increasing \( \text{key}(H[i]) \) to \( \alpha \) makes \( H \) a heap.
- Property is **local**: \( H \) is a heap except “around” \( H[i] \).
- **Prove by induction on** \( i \).
Correctness of Heapify-up

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- **Base case**: $i = 1$.

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- **Property is local**: $H$ is a heap except “around” $H[i]$.

- **Prove by induction on $i$**.

- **Base case**: $i = 1$.
  - $H$ is a heap with key of $H[1]$ too small.
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- Base case: \( i = 1 \).
  - \( H \) is a heap with key of \( H[1] \) too small.
  - There is a value \( \alpha \geq \text{key}(H[1]) \) such that increasing \( \text{key}(H[1]) \) to \( \alpha \) makes \( H \) a heap.
  - \( \text{key}(H[1]) \leq \alpha \leq \text{key}(H[2]), \text{key}(H[3]) \implies H \) is a heap.
**Correctness of Heapify-up**

- **Property is local:** $H$ is a heap except “around” $H[i]$.
- **Prove by induction on $i$.**
- **Inductive hypothesis:** If $H$ is heap with the key of $H[\lfloor i/2 \rfloor]$ too small, then $\text{Heapify-up}(H, \lfloor i/2 \rfloor)$ creates a heap.

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- **Inductive step:**
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- Property is **local**: \( H \) is a heap except “around” \( H[i] \).
- Prove by induction on \( i \).
- Inductive hypothesis: If \( H \) is heap with the key of \( H[\lfloor i/2 \rfloor] \) too small, then \( \text{Heapify-up}(H, \lfloor i/2 \rfloor) \) creates a heap.
- Inductive step: After the swap statement in \( \text{Heapify-up}(H, i) \), \( H \) is a heap or almost a heap with the key of \( H[\lfloor i/2 \rfloor] \) too small.
Running time of Heapify-up

Heapify-up(H,i):
    If $i > 1$ then
        let $j = \text{parent}(i) = \lfloor i/2 \rfloor$
        If key[H[i]] < key[H[j]] then
            swap the array entries H[i] and H[j]
            Heapify-up(H,j)
        Endif
    Endif
Endif

Running time of Heapify-up(i)
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swap the array entries H[i] and H[j]

Heapify-up(H,j)

Endif

Endif

Running time of Heapify-up($i$) is $O(\log i)$.

$$T(i) \leq \begin{cases} T(\lfloor i/2 \rfloor) + O(1) & \text{if } i > 1 \\ O(1) & \text{if } i = 1 \end{cases}$$
Deleting an Element: Heapify-down

1. Suppose $H$ has $n + 1$ elements.
3. If element at $H[i]$ is too small, fix heap order using $\text{Heapify-up}(H, i)$.
4. If element at $H[i]$ is too large, fix heap order using $\text{Heapify-down}(H, i)$.

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**Heapify-down($H, i$):**

Let $n = \text{length}(H)$

If $2i > n$ then
   Terminate with $H$ unchanged
Else if $2i < n$ then
   Let $\text{left} = 2i$, and $\text{right} = 2i + 1$
   Let $j$ be the index that minimizes $\text{key}[H[\text{left}]]$ and $\text{key}[H[\text{right}]]$
Else if $2i = n$ then
   Let $j = 2i$
Endif

If $\text{key}[H[j]] < \text{key}[H[i]]$ then
   swap the array entries $H[i]$ and $H[j]$
   $\text{Heapify-down}(H, j)$
Endif
**Example of Heapify-down**

The **Heapify-down** process is moving element \( w \) down, toward the leaves.

**Figure 2.5** The Heapify-down process: Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).
Correctness of Heapify-down

The Heapify-down process is moving element $w$ down, toward the leaves.

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Correctness of Heapify-down

$H$ is *almost a heap with key of $H[i]$ too big* if there is a value $\alpha \leq \text{key}(H[i])$ such that decreasing $\text{key}(H[i])$ to $\alpha$ makes $H$ a heap.
Correctness of \textbf{Heapify-down}

- \textit{Proof by reverse induction on} \( i \).
- \textit{Base case}:

\( H \) is \textit{almost a heap with key of} \( H[i] \) \textit{too big} if there is a value \( \alpha \leq \text{key}(H[i]) \) such that decreasing \( \text{key}(H[i]) \) to \( \alpha \) makes \( H \) a heap.
Correctness of Heapify-down

- **H** is *almost a heap with key of \( H[i] \) too big* if there is a value \( \alpha \leq \text{key}(H[i]) \) such that decreasing \( \text{key}(H[i]) \) to \( \alpha \) makes \( H \) a heap.

- Proof by reverse induction on \( i \).

- Base case: \( 2i > n \).

- Inductive hypothesis: If \( H \) almost a heap with the key of \( H[2i] \) or \( H[2i + 1] \) too big, then Heapify-down\((H, 2i)\) or Heapify-down\((H, 2i + 1)\) creates a heap.

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Correctness of Heapify-down

- $H$ is **almost a heap with key of $H[i]$ too big** if there is a value $\alpha \leq \text{key}(H[i])$ such that decreasing $\text{key}(H[i])$ to $\alpha$ makes $H$ a heap.
- Proof by **reverse induction** on $i$.
- Base case: $2i > n$.
- Inductive hypothesis: If $H$ almost a heap with the key of $H[2i]$ or $H[2i + 1]$ too big, then Heapify-down($H, 2i$) or Heapify-down($H, 2i + 1$) creates a heap.
- Inductive step:
Correctness of Heapify-down

- **$H$ is almost a heap with key of $H[i]$ too big** if there is a value $\alpha \leq \text{key}(H[i])$ such that decreasing $\text{key}(H[i])$ to $\alpha$ makes $H$ a heap.
- **Proof by reverse induction** on $i$.
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- **Inductive step**: After the swap statement in Heapify-down($H, i$), $H$ is a heap or almost a heap with the key of $H[2i]$ or $H[2i + 1]$ too big.

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Heapify-down(H, i):
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   Else if \( 2i < n \) then
      Let left = \( 2i \), and right = \( 2i + 1 \)
      Let \( j \) be the index that minimizes \( \text{key}[H[\text{left}]] \) and \( \text{key}[H[\text{right}]] \)
   Else if \( 2i = n \) then
      Let \( j = 2i \)
   Endif
   If \( \text{key}[H[j]] < \text{key}[H[i]] \) then
      swap the array entries \( H[i] \) and \( H[j] \)
      Heapify-down(H, j)
   Endif

Running time of Heapify-down(H, i)
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   Endif

Running time of Heapify-down(H, i) is $O(\log n/i)$.

\[
T(i) = \begin{cases} 
\max (T(2i), T(2i+1)) + 1 & \text{if } i > 1 \\
O(1) & \text{if } 2i > n
\end{cases}
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Final algorithm:

- **Insert** each number in a priority queue $H$.
- **Repeatedly** find the smallest number in $H$, output it, and delete it from $H$.

Each insertion and deletion takes $O(\log n)$ time for a total running time of $O(n \log n)$. 

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