Applications of Network Flow

T. M. Murali

April 7, 9, 2009
Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.

We will only sketch proofs. Read details from the textbook.
Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.
Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.
- We will only sketch proofs. Read details from the textbook.
Matching in Bipartite Graphs

- **Bipartite Graph**: a graph $G(V, E)$ where
  1. $V = X \cup Y$, $X$ and $Y$ are disjoint and
  2. $E \subseteq X \times Y$.

- Bipartite graphs model situations in which objects are matched with or assigned to other objects: e.g., marriages, residents/hospitals, jobs/machines.
Matching in Bipartite Graphs

- **Bipartite Graph**: a graph \( G(V, E) \) where
  1. \( V = X \cup Y \), \( X \) and \( Y \) are disjoint and
  2. \( E \subseteq X \times Y \).

- Bipartite graphs model situations in which objects are matched with or assigned to other objects: e.g., marriages, residents/hospitals, jobs/machines.

- A **matching** in a bipartite graph \( G \) is a set \( M \subseteq E \) of edges such that each node of \( V \) is incident on at most one edge of \( M \).

- A set of edges \( M \) is a **perfect matching** if every node in \( V \) is incident on exactly one edge in \( M \).
Bipartite Graph Matching Problem

**BIPARTITE MATCHING**

**INSTANCE:** A Bipartite graph $G$.

**SOLUTION:** The matching of largest size in $G$. 
Algorithm for Bipartite Graph Matching

Convert $G$ to a flow network $G'$: direct edges from $X$ to $Y$, add nodes $s$ and $t$, connect $s$ to each node in $X$, connect each node in $Y$ to $t$, set all edge capacities to 1.

Compute the maximum flow in $G'$.

Claim: the value of the maximum flow is the size of the maximum matching.

Figure 7.9 (a) A bipartite graph. (b) The corresponding flow network, with all capacities equal to 1.
Correctness of Bipartite Graph Matching Algorithm

- Matching $\rightarrow$ flow: if there is a matching with $k$ edges in $G$, there is an $s-t$ flow of value $k$ in $G'$. 

- Flow $\rightarrow$ matching: if there is an integer-valued flow $f'$ in $G'$ with value $k$, there is a matching $M$ in $G$ with $k$ edges.

- There is an integer-valued flow $f$ of value $k$ $\Rightarrow$ flow along any edge is 0 or 1.

- Let $M$ be the set of edges not incident on $s$ or $t$ with flow equal to 1.

- Claim: $M$ contains $k$ edges.

- Claim: Each node in $X$ (respectively, $Y$) is the tail (respectively, head) of at most one edge in $M$.

- Conclusion: size of the maximum matching in $G$ is equal to the value of the maximum flow in $G'$; the edges in this matching are those that carry flow from $X$ to $Y$ in $G'$.

- Read the book on what augmenting paths mean in this context.
Correctness of Bipartite Graph Matching Algorithm

- Matching $\rightarrow$ flow: if there is a matching with $k$ edges in $G$, there is an $s$-$t$ flow of value $k$ in $G'$.
- Flow $\rightarrow$ matching: if there is an integer-valued flow $f'$ in $G'$ with value $k$, there is a matching $M$ in $G$ with $k$ edges.
Correctness of Bipartite Graph Matching Algorithm

- **Matching → flow**: if there is a matching with $k$ edges in $G$, there is an $s$-$t$ flow of value $k$ in $G'$.

- **Flow → matching**: if there is an integer-valued flow $f'$ in $G'$ with value $k$, there is a matching $M$ in $G$ with $k$ edges.
  - There is an integer-valued flow $f$ of value $k$ ⇒ flow along any edge is 0 or 1.
  - Let $M$ be the set of edges not incident on $s$ or $t$ with flow equal to 1.
**Correctness of Bipartite Graph Matching Algorithm**

- **Matching → flow:** if there is a matching with \( k \) edges in \( G \), there is an \( s-t \) flow of value \( k \) in \( G' \).

- **Flow → matching:** if there is an integer-valued flow \( f' \) in \( G' \) with value \( k \), there is a matching \( M \) in \( G \) with \( k \) edges.
  - There is an integer-valued flow \( f \) of value \( k \) \( \Rightarrow \) flow along any edge is 0 or 1.
  - Let \( M \) be the set of edges not incident on \( s \) or \( t \) with flow equal to 1.
  - **Claim:** \( M \) contains \( k \) edges.
Correctness of Bipartite Graph Matching Algorithm

- Matching → flow: if there is a matching with \( k \) edges in \( G \), there is an \( s-t \) flow of value \( k \) in \( G' \).
- Flow → matching: if there is an integer-valued flow \( f' \) in \( G' \) with value \( k \), there is a matching \( M \) in \( G \) with \( k \) edges.
  - There is an integer-valued flow \( f \) of value \( k \) ⇒ flow along any edge is 0 or 1.
  - Let \( M \) be the set of edges not incident on \( s \) or \( t \) with flow equal to 1.
  - Claim: \( M \) contains \( k \) edges.
  - Claim: Each node in \( X \) (respectively, \( Y \)) is the tail (respectively, head) of at most one edge in \( M \).
Correctness of Bipartite Graph Matching Algorithm

▶ Matching $\rightarrow$ flow: if there is a matching with $k$ edges in $G$, there is an $s$-$t$ flow of value $k$ in $G'$.

▶ Flow $\rightarrow$ matching: if there is an integer-valued flow $f'$ in $G'$ with value $k$, there is a matching $M$ in $G$ with $k$ edges.
  ▶ There is an integer-valued flow $f$ of value $k$ $\Rightarrow$ flow along any edge is 0 or 1.
  ▶ Let $M$ be the set of edges not incident on $s$ or $t$ with flow equal to 1.
  ▶ Claim: $M$ contains $k$ edges.
  ▶ Claim: Each node in $X$ (respectively, $Y$) is the tail (respectively, head) of at most one edge in $M$.

▶ Conclusion: size of the maximum matching in $G$ is equal to the value of the maximum flow in $G'$; the edges in this matching are those that carry flow from $X$ to $Y$ in $G'$. 

Read the book on what augmenting paths mean in this context.
Correctness of Bipartite Graph Matching Algorithm

- Matching → flow: if there is a matching with \( k \) edges in \( G \), there is an \( s-t \) flow of value \( k \) in \( G' \).
- Flow → matching: if there is an integer-valued flow \( f' \) in \( G' \) with value \( k \), there is a matching \( M \) in \( G \) with \( k \) edges.
  - There is an integer-valued flow \( f \) of value \( k \) ⇒ flow along any edge is 0 or 1.
  - Let \( M \) be the set of edges not incident on \( s \) or \( t \) with flow equal to 1.
  - Claim: \( M \) contains \( k \) edges.
  - Claim: Each node in \( X \) (respectively, \( Y \)) is the tail (respectively, head) of at most one edge in \( M \).

- Conclusion: size of the maximum matching in \( G \) is equal to the value of the maximum flow in \( G' \); the edges in this matching are those that carry flow from \( X \) to \( Y \) in \( G' \).
- Read the book on what augmenting paths mean in this context.
Running time of Bipartite Graph Matching Algorithm

- Suppose $G$ has $m$ edges and $n$ nodes in $X$ and in $Y$. 
Running time of Bipartite Graph Matching Algorithm

- Suppose $G$ has $m$ edges and $n$ nodes in $X$ and in $Y$.
- $C \leq n$.
- Ford-Fulkerson algorithm runs in $O(mn)$ time.
- How long does the scaling algorithm take?
Running time of Bipartite Graph Matching Algorithm

- Suppose $G$ has $m$ edges and $n$ nodes in $X$ and in $Y$.
- $C \leq n$.
- Ford-Fulkerson algorithm runs in $O(mn)$ time.
- How long does the scaling algorithm take? $O(m^2 \log n)$ time.
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching?
How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
- $G$ has no perfect matching iff
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
- $G$ has no perfect matching iff the maximum capacity of a cut in $G'$ is less than $n$. Therefore, the cut is a certificate.
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
- $G$ has no perfect matching iff the maximum capacity of a cut in $G'$ is less than $n$. Therefore, the cut is a certificate.
- But we would like the certificate in terms of $G$. 

Hall’s Theorem: Let $G(X \cup Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then $G$ either has a perfect matching or there is a subset $A \subseteq X$ such that $|A| > |\Gamma(A)|$. A perfect matching or such a subset can be computed in $O(mn)$ time. Read proof in the textbook.
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
- $G$ has no perfect matching iff the maximum capacity of a cut in $G'$ is less than $n$. Therefore, the cut is a certificate.
- But we would like the certificate in terms of $G$.
  - For example, two nodes in $X$ with one incident edge each with the same neighbour in $Y$. 
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?
- What can such certificates look like?
- $G$ has no perfect matching iff the maximum capacity of a cut in $G'$ is less than $n$. Therefore, the cut is a certificate.
- But we would like the certificate in terms of $G$.
  - For example, two nodes in $X$ with one incident edge each with the same neighbour in $Y$.
  - Generally, a subset $A \subseteq X$ with neighbours $\Gamma(A) \subseteq Y$, such that $|A| > |\Gamma(A)|$. 

Hall’s Theorem: Let $G( X \cup Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then $G$ either has a perfect matching or there is a subset $A \subseteq X$ such that $|A| > |\Gamma(A)|$. A perfect matching or such a subset can be computed in $O(mn)$ time. Read proof in the textbook.
Bipartite Graphs without Perfect Matchings

How do we determine if a bipartite graph $G$ has a perfect matching? Find the maximum matching and check if it is perfect.

Suppose $G$ has no perfect matching. Can we exhibit a short “certificate” of that fact?

What can such certificates look like?

$G$ has no perfect matching iff the maximum capacity of a cut in $G'$ is less than $n$. Therefore, the cut is a certificate.

But we would like the certificate in terms of $G$.

For example, two nodes in $X$ with one incident edge each with the same neighbour in $Y$.

Generally, a subset $A \subseteq X$ with neighbours $\Gamma(A) \subseteq Y$, such that $|A| > |\Gamma(A)|$.

Hall’s Theorem: Let $G(X \cup Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then $G$ either has a perfect matching or there is a subset $A \subseteq X$ such that $|A| > |\Gamma(A)|$. A perfect matching or such a subset can be computed in $O(mn)$ time.
Bipartite Graphs without Perfect Matchings

- How do we determine if a bipartite graph \( G \) has a perfect matching? Find the maximum matching and check if it is perfect.

- Suppose \( G \) has no perfect matching. Can we exhibit a short “certificate” of that fact?

- What can such certificates look like?

- \( G \) has no perfect matching iff the maximum capacity of a cut in \( G' \) is less than \( n \). Therefore, the cut is a certificate.

- But we would like the certificate in terms of \( G \).
  - For example, two nodes in \( X \) with one incident edge each with the same neighbour in \( Y \).
  - Generally, a subset \( A \subseteq X \) with neighbours \( \Gamma(A) \subseteq Y \), such that \(|A| > |\Gamma(A)|\).

- **Hall’s Theorem**: Let \( G(X \cup Y, E) \) be a bipartite graph such that \(|X| = |Y|\). Then \( G \) either has a perfect matching or there is a subset \( A \subseteq X \) such that \(|A| > |\Gamma(A)|\). A perfect matching or such a subset can be computed in \( O(mn) \) time. Read proof in the textbook.
Edge-Disjoint Paths

- A set of paths in a graph $G$ is *edge disjoint* if each edge in $G$ appears in at most one path.
Edge-Disjoint Paths

- A set of paths in a graph $G$ is *edge disjoint* if each edge in $G$ appears in at most one path.

**Directed Edge-Disjoint Paths**

**INSTANCE:** Directed graph $G(V, E)$ with two distinguished nodes $s$ and $t$.

**SOLUTION:** The maximum number of edge-disjoint paths between $s$ and $t$. 
Mapping to the Max-Flow Problem

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
Mapping to the Max-Flow Problem

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
- Paths $\rightarrow$ flow: if there are $k$ edge-disjoint paths from $s$ to $t$, send one unit of flow along each to yield a flow with value $k$.
- Flow $\rightarrow$ paths: Suppose there is an integer-valued flow of value $k$. Are there $k$ edge-disjoint paths? If so, what are they?
Mapping to the Max-Flow Problem

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
- Paths $\rightarrow$ flow: if there are $k$ edge-disjoint paths from $s$ to $t$, send one unit of flow along each to yield a flow with value $k$.
- Flow $\rightarrow$ paths: Suppose there is an integer-valued flow of value $k$. Are there $k$ edge-disjoint paths? If so, what are they?
- Construct $k$ edge-disjoint paths from a flow of value $\geq k$.
  - There is an integral flow. Therefore, flow on each edge is 0 or 1.
**Mapping to the Max-Flow Problem**

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
- Paths $\rightarrow$ flow: if there are $k$ edge-disjoint paths from $s$ to $t$, send one unit of flow along each to yield a flow with value $k$.
- Flow $\rightarrow$ paths: Suppose there is an integer-valued flow of value $k$. Are there $k$ edge-disjoint paths? If so, what are they?
- Construct $k$ edge-disjoint paths from a flow of value $\geq k$.
  - There is an integral flow. Therefore, flow on each edge is 0 or 1.
  - Claim: if $f$ is a 0-1 valued flow of value $\nu$, then the set of edges with flow $f(e) = 1$ contains a set of $\nu$ edge-disjoint paths.
Mapping to the Max-Flow Problem

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
- Paths $\rightarrow$ flow: if there are $k$ edge-disjoint paths from $s$ to $t$, send one unit of flow along each to yield a flow with value $k$.
- Flow $\rightarrow$ paths: Suppose there is an integer-valued flow of value $k$. Are there $k$ edge-disjoint paths? If so, what are they?
- Construct $k$ edge-disjoint paths from a flow of value $\geq k$.
  - There is an integral flow. Therefore, flow on each edge is 0 or 1.
  - Claim: if $f$ is a 0-1 valued flow of value $\nu$, then the set of edges with flow $f(e) = 1$ contains a set of $\nu$ edge-disjoint paths.
  - Prove by induction on the number of edges in $f$ that carry flow.
Mapping to the Max-Flow Problem

- Convert $G$ into a flow network: $s$ is the source, $t$ is the sink, each edge has capacity 1.
- Paths $\rightarrow$ flow: if there are $k$ edge-disjoint paths from $s$ to $t$, send one unit of flow along each to yield a flow with value $k$.
- Flow $\rightarrow$ paths: Suppose there is an integer-valued flow of value $k$. Are there $k$ edge-disjoint paths? If so, what are they?
- Construct $k$ edge-disjoint paths from a flow of value $\geq k$.
  - There is an integral flow. Therefore, flow on each edge is 0 or 1.
  - Claim: if $f$ is a 0-1 valued flow of value $\nu$, then the set of edges with flow $f(e) = 1$ contains a set of $\nu$ edge-disjoint paths.
  - Prove by induction on the number of edges in $f$ that carry flow.
- We just proved: there are $k$ edge-disjoint paths from $s$ to $t$ in a directed graph $G$ iff the maximum value of an $s$-$t$ flow in $G$ is $\geq k$. 
Running Time of the Edge-Disjoint Paths Algorithm

- Given a flow of value $k$, how quickly can we determine the $k$ edge-disjoint paths?
Running Time of the Edge-Disjoint Paths Algorithm

- Given a flow of value $k$, how quickly can we determine the $k$ edge-disjoint paths? $O(mn)$ time.
- Corollary: The Ford-Fulkerson algorithm can be used to find a maximum set of edge-disjoint $s-t$ paths in a directed graph $G$ in
Running Time of the Edge-Disjoint Paths Algorithm

- Given a flow of value $k$, how quickly can we determine the $k$ edge-disjoint paths? $O(mn)$ time.
- Corollary: The Ford-Fulkerson algorithm can be used to find a maximum set of edge-disjoint $s$-$t$ paths in a directed graph $G$ in $O(mn)$ time.
Certificate for Edge-Disjoint Paths Algorithm

- A set $F \subseteq E$ of edge separates $s$ and $t$ if the graph $(V, E - F)$ contains no $s$-$t$ paths.
Certificate for Edge-Disjoint Paths Algorithm

- A set $F \subseteq E$ of edge separates $s$ and $t$ if the graph $(V, E - F)$ contains no $s$-$t$ paths.

- **Menger’s Theorem**: In every directed graph with nodes $s$ and $t$, the maximum number of edge-disjoint $s$-$t$ paths is equal to the minimum number of edges whose removal disconnects $s$ from $t$. 
Can extend the theorem to *undirected* graphs.
Can extend the theorem to *undirected* graphs.

Replace each edge with two directed edges of capacity 1 and apply the algorithm for directed graphs.
Edge-Disjoint Paths in Undirected Graphs

- Can extend the theorem to \textit{undirected} graphs.
- Replace each edge with two directed edges of capacity 1 and apply the algorithm for directed graphs.
- Problem: Both counterparts of an undirected edge \((u, v)\) may be used by different edge-disjoint paths in the directed graph.
Edge-Disjoint Paths in Undirected Graphs

- Can extend the theorem to *undirected* graphs.
- Replace each edge with two directed edges of capacity 1 and apply the algorithm for directed graphs.
- Problem: Both counterparts of an undirected edge \((u, v)\) may be used by different edge-disjoint paths in the directed graph.
- Can obtain an integral flow where only one of the directed counterparts of \((u, v)\) has non-zero flow.
Edge-Disjoint Paths in Undirected Graphs

- Can extend the theorem to *undirected* graphs.
- Replace each edge with two directed edges of capacity 1 and apply the algorithm for directed graphs.
- Problem: Both counterparts of an undirected edge \((u, v)\) may be used by different edge-disjoint paths in the directed graph.
- Can obtain an integral flow where only one of the directed counterparts of \((u, v)\) has non-zero flow.
- We can find the maximum number of edge-disjoint paths in \(O(mn)\) time.
- We can prove a version of Menger’s theorem for undirected graphs: in every undirected graph with nodes \(s\) and \(t\), the maximum number of edge-disjoint \(s-t\) paths is equal to the minimum number of edges whose removal separates \(s\) from \(t\).
Extension of Max-Flow Problem

- Suppose we have a set $S$ of multiple sources and a set $T$ of multiple sinks.
- Each source can send flow to any sink.
- Let us not maximise flow here but formulate the problem in terms of demands and supplies.
Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:
Circulation with Demands

We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:

- $d_v > 0$: node is a sink, it has a “demand” for $d_v$ units of flow.
- $d_v < 0$: node is a source, it has a “supply” of $-d_v$ units of flow.
- $d_v = 0$: node simply receives and transmits flow.
Circulation with Demands

We are given a graph $G(V, E)$ with capacity function $c : E \to \mathbb{Z}^+$ and a demand function $d : V \to \mathbb{Z}$:

- $d_v > 0$: node is a sink, it has a “demand” for $d_v$ units of flow.
- $d_v < 0$: node is a source, it has a “supply” of $-d_v$ units of flow.
- $d_v = 0$: node simply receives and transmits flow.
- $S$ is the set of nodes with negative demand and $T$ is the set of nodes with positive demand.
Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:
  - $d_v > 0$: node is a sink, it has a “demand” for $d_v$ units of flow.
  - $d_v < 0$: node is a source, it has a “supply” of $-d_v$ units of flow.
  - $d_v = 0$: node simply receives and transmits flow.
  - $S$ is the set of nodes with negative demand and $T$ is the set of nodes with positive demand.

- A circulation with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
Circulation with Demands

We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:

- $d_v > 0$: node is a sink, it has a “demand” for $d_v$ units of flow.
- $d_v < 0$: node is a source, it has a “supply” of $-d_v$ units of flow.
- $d_v = 0$: node simply receives and transmits flow.
- $S$ is the set of nodes with negative demand and $T$ is the set of nodes with positive demand.

A circulation with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies

(i) (Capacity conditions) For each $e \in E$, $0 \leq f(e) \leq c(e)$.
(ii) (Demand conditions) For each node $v$, $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$. 

- INSTANCE:
  A directed graph $G(V, E)$, $c : E \rightarrow \mathbb{Z}^+$, and $d : V \rightarrow \mathbb{Z}$.

- SOLUTION:
  Does there exist a circulation that is feasible, i.e., it meets the capacity and demand conditions?
Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:
  - $d_v > 0$: node is a sink, it has a “demand” for $d_v$ units of flow.
  - $d_v < 0$: node is a source, it has a “supply” of $-d_v$ units of flow.
  - $d_v = 0$: node simply receives and transmits flow.
  - $S$ is the set of nodes with negative demand and $T$ is the set of nodes with positive demand.

- A circulation with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
  1. (Capacity conditions) For each $e \in E$, $0 \leq f(e) \leq c(e)$.
  2. (Demand conditions) For each node $v$, $f^\text{in}(v) - f^\text{out}(v) = d_v$.

Circulation with Demands

INSTANCE: A directed graph $G(V, E)$, $c : E \rightarrow \mathbb{Z}^+$, and $d : V \rightarrow \mathbb{Z}$.

SOLUTION: Does there exist a circulation that is feasible, i.e., it meets the capacity and demand conditions?
Properties of Feasible Circulations

- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$. 


Properties of Feasible Circulations

- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$.
- Corollary: $\sum_{v, d_v > 0} d_v = \sum_{v, d_v < 0} -d_v$. Let $D$ denote this common value.
Create a new graph $G' = G$ and

1. create two new nodes in $G'$: a source $s^*$ and a sink $t^*$;
2. connect $s^*$ to each node $v$ in $S$ using an edge with capacity $-d_v$;
3. connect each node $v$ in $T$ to $t^*$ using an edge with capacity $d_v$. 

Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.

Figure 7.13 (a) An instance of the Circulation Problem together with a solution: Numbers inside the nodes are demands; numbers labeling the edges are capacities and flow values, with the flow values inside boxes. (b) The result of reducing this instance to an equivalent instance of the Maximum-Flow Problem.
Computing a Feasible Circulation

We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f)$

Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.
Computing a Feasible Circulation

We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.
We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.

Circulation $\rightarrow$ flow.
Computing a Feasible Circulation

- We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.
- Circulation $\rightarrow$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge $(s^*, v)$ and $d_v$ units of flow along each edge $(v, t^*)$. The value of this flow is $D$.
We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.

Circulation $\rightarrow$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge $(s^*, v)$ and $d_v$ units of flow along each edge $(v, t^*)$. The value of this flow is $D$.

Flow $\rightarrow$ circulation. If there is an $s$-$t$ flow of value $D$ in $G'$,
Computing a Feasible Circulation

We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.

Circulation $\rightarrow$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge $(s^*, v)$ and $d_v$ units of flow along each edge $(v, t^*)$. The value of this flow is $D$.

Flow $\rightarrow$ circulation. If there is an $s$-$t$ flow of value $D$ in $G'$, edges incident on $s^*$ and on $t^*$ must be saturated with flow. Deleting these edges from $G'$ yields a feasible circulation in $G$. 

Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.
Computing a Feasible Circulation

We will look for a maximum $s$-$t$ flow $f$ in $G'$; $\nu(f) \leq D$.

Circulation $\to$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge $(s^*, v)$ and $d_v$ units of flow along each edge $(v, t^*)$. The value of this flow is $D$.

Flow $\to$ circulation. If there is an $s$-$t$ flow of value $D$ in $G'$, edges incident on $s^*$ and on $t^*$ must be saturated with flow. Deleting these edges from $G'$ yields a feasible circulation in $G$.

We have just proved that there is a feasible circulation with demands in $G$ iff the maximum $s$-$t$ flow in $G'$ has value $D$. 

Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.
Circulation with Demands and Lower Bounds

- We want to force the flow to use certain edges.

We want to force the flow to use certain edges.

We are given a graph $G(V,E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand $d_v$ on each vertex.

A circulation with demands is a function $f: E \rightarrow \mathbb{R}^+$ that satisfies:

(i) (Capacity conditions) For each $e \in E$, $l(e) \leq f(e) \leq c(e)$.

(ii) (Demand conditions) For each node $v$, $f_{in}(v) - f_{out}(v) = d_v$.

Is there a feasible circulation?
Circulation with Demands and Lower Bounds

- We want to force the flow to use certain edges.
- We are given a graph \( G(V, E) \) with a capacity \( c(e) \) and a lower bound \( 0 \leq l(e) \leq c(e) \) on each edge and a demand \( d_v \) on each vertex.
Circulation with Demands and Lower Bounds

- We want to force the flow to use certain edges.
- We are given a graph $G(V, E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand $d_v$ on each vertex.
- A \textit{circulation} with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
We want to force the flow to use certain edges.

We are given a graph $G(V, E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand $d_v$ on each vertex.

A circulation with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies

(i) (Capacity conditions) For each $e \in E$, $l(e) \leq f(e) \leq c(e)$.

(ii) (Demand conditions) For each node $v$, $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$. 

Is there a feasible circulation?
Circulation with Demands and Lower Bounds

- We want to force the flow to use certain edges.
- We are given a graph $G(V, E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand $d_v$ on each vertex.
- A *circulation* with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
  1. *(Capacity conditions)* For each $e \in E$, $l(e) \leq f(e) \leq c(e)$.
  2. *(Demand conditions)* For each node $v$, $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$.
- Is there a feasible circulation?
Algorithm for Circulation with Lower Bounds

▶ Strategy is to reduce the problem to one with no lower bounds on edges.
Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
- Demand conditions may be violated. Let $L_v = f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} l(e) - \sum_{e \text{ out of } v} l(e)$. 


Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
- Demand conditions may be violated. Let $L_v = f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} l(e) - \sum_{e \text{ out of } v} l(e)$.
- If $L_v \neq d_v$, we can superimpose a circulation $f_1$ on top of $f_0$ such that $f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$. 

How much capacity do we have left on each edge? $c(e) - l(e)$. 

Approach: define a new graph $G'$ with the same nodes and edges: lower bound on each edge is 0, capacity of edge $e$ is $c(e) - l(e)$, and demand of node $v$ is $d_v - L_v$. 

Claim: there is a feasible circulation in $G$ iff there is a feasible circulation in $G'$. 
Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
- Demand conditions may be violated. Let $L_v = f_{0in}(v) - f_{0out}(v) = \sum_{e \text{ into } v} l(e) - \sum_{e \text{ out of } v} l(e)$.
- If $L_v \neq d_v$, we can superimpose a circulation $f_1$ on top of $f_0$ such that $f_{1in}(v) - f_{1out}(v) = d_v - L_v$.
- How much capacity do we have left on each edge?
Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
- Demand conditions may be violated. Let
  \[ L_v = f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} l(e) - \sum_{e \text{ out of } v} l(e). \]
- If $L_v \neq d_v$, we can superimpose a circulation $f_1$ on top of $f_0$ such that $f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$.
- How much capacity do we have left on each edge? $c(e) - l(e)$.
Algorithm for Circulation with Lower Bounds

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation $f_0$ that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?
- Demand conditions may be violated. Let $L_v = f_{0}^{\text{in}}(v) - f_{0}^{\text{out}}(v) = \sum_{e \text{ into } v} l(e) - \sum_{e \text{ out of } v} l(e)$.
- If $L_v \neq d_v$, we can superimpose a circulation $f_1$ on top of $f_0$ such that $f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$.
- How much capacity do we have left on each edge? $c(e) - l(e)$.
- Approach: define a new graph $G'$ with the same nodes and edges: lower bound on each edge is 0, capacity of edge $e$ is $c(e) - l(e)$, and demand of node $v$ is $d_v - L_v$.
- Claim: there is a feasible circulation in $G$ iff there is a feasible circulation in $G'$. 
Data Mining

- Algorithmic study of unexpected patterns in large quantities of data.
- Study customer preferences is an important topic.
  - Customers who buy diapers also buy beer:
  - People who bought “Harry Potter and the Deathly Hallows” also bought “Making Money (Discworld)”.
- Store cards allow companies to keep track of your history of shopping.
Survey Design

- Company sells $k$ products.
- Company has a database of purchase histories of many customers.
- Company wants to send a customised survey to each of its $n$ customers to further understand their preferences.

Is it possible to design a survey that satisfies these constraints?
Survey Design

- Company sells $k$ products.
- Company has a database of purchase histories of many customers.
- Company wants to send a customised survey to each of its $n$ customers to further understand their preferences.
- Survey must satisfy certain constraints:
  1. Each customer receives questions about a subset of products.
  2. A customer receives questions only about products he/she has bought.
  3. The questionnaire must be informative but not too long: each customer $i$ should be asked about a number of products between $c_i$ and $c_i'$.
  4. Each product must have enough data collected: between $p_j$ and $p_j'$ customers should be asked about product $j$. 

Is it possible to design a survey that satisfies these constraints?
Survey Design

- Company sells $k$ products.
- Company has a database of purchase histories of many customers.
- Company wants to send a customised survey to each of its $n$ customers to further understand their preferences.
- Survey must satisfy certain constraints:
  1. Each customer receives questions about a subset of products.
  2. A customer receives questions only about products he/she has bought.
  3. The questionnaire must be informative but not too long: each customer $i$ should be asked about a number of products between $c_i$ and $c'_i$.
  4. Each product must have enough data collected: between $p_j$ and $p'_j$ customers should be asked about product $j$.
- Is it possible to design a survey that satisfies this constraints?
Formalising the Survey Design Problem

- Input is a bipartite graph $G$:
  - Nodes are $n$ customers and $k$ products.
  - There is an edge between customer $i$ and product $j$ iff the customer has purchased the product at some time.
  - For each customer $1 \leq i \leq n$, limits $c_i \leq c_i'$ on the number of products he or she can be asked about.
  - For each product $1 \leq j \leq k$, limits $p_j \leq p_j'$ on the number of distinct customers asked about the product.
Solving the Survey Design Problem

- Reduce the problem to a circulation problem on a flow network $G'$ with demands and lower bounds (lbs).
Solving the Survey Design Problem

- Reduce the problem to a circulation problem on a flow network $G'$ with demands and lower bounds (lbs).

- Orient edges in $G$ from customers to products: capacity 1, lb 0.

- Add node $s$, edges $(s, i)$ to each customer: capacity $c'_i$, lb $c_i$.

- Add node $t$, edges $(j, t)$ from each product: capacity $p'_i$, lb $p_i$.

- Set node demands to

Figure 7.16 The Survey Design Problem can be reduced to the problem of finding a feasible circulation: Flow passes from customers (with capacity bounds indicating how many questions they can be asked) to products (with capacity bounds indicating how many questions should be asked about each product).
Solving the Survey Design Problem

- Reduce the problem to a circulation problem on a flow network $G'$ with demands and lower bounds (lbs).

- Orient edges in $G$ from customers to products: capacity 1, lb 0.
- Add node $s$, edges $(s, i)$ to each customer: capacity $c'_i$, lb $c_i$.
- Add node $t$, edges $(j, t)$ from each product: capacity $p'_i$, lb $p_i$.
- Set node demands to 0.

Figure 7.16 The Survey Design Problem can be reduced to the problem of finding a feasible circulation: Flow passes from customers (with capacity bounds indicating how many questions they can be asked) to products (with capacity bounds indicating how many questions should be asked about each product).
Solving the Survey Design Problem

- Reduce the problem to a circulation problem on a flow network $G'$ with demands and lower bounds (lbs).
- Orient edges in $G$ from customers to products: capacity 1, lb 0.
- Add node $s$, edges $(s, i)$ to each customer: capacity $c'_i$, lb $c_i$.
- Add node $t$, edges $(j, t)$ from each product: capacity $p'_i$, lb $p_i$.
- Set node demands to 0.
- Add edge from $t$ to $s$: capacity $\sum_i c'_i$, lb $\sum_i c_i$.

Figure 7.16 The Survey Design Problem can be reduced to the problem of finding a feasible circulation: Flow passes from customers (with capacity bounds indicating how many questions they can be asked) to products (with capacity bounds indicating how many questions should be asked about each product).
Solving the Survey Design Problem

- Reduce the problem to a circulation problem on a flow network $G'$ with demands and lower bounds (lbs).

- Orient edges in $G$ from customers to products: capacity 1, lb 0.

- Add node $s$, edges $(s, i)$ to each customer: capacity $c'_i$, lb $c_i$.

- Add node $t$, edges $(j, t)$ from each product: capacity $p'_i$, lb $p_i$.

- Set node demands to 0.

- Add edge from $t$ to $s$: capacity $\sum_i c'_i$, lb $\sum_i c_i$.

- Claim: $G'$ has a feasible circulation iff there is a feasible survey.

Figure 7.16 The Survey Design Problem can be reduced to the problem of finding a feasible circulation: Flow passes from customers (with capacity bounds indicating how many questions they can be asked) to products (with capacity bounds indicating how many questions should be asked about each product).
Image Segmentation

- A fundamental problem in computer vision is that of segmenting an image into coherent regions.

- A basic segmentation problem is that of partitioning an image into a foreground and a background: label each pixel in the image as belonging to the foreground or the background.
Formulating the Image Segmentation Problem

- Let $V$ be the set of pixels in an image.
- Let $E$ be the set of pairs of neighbouring pixels.
- $V$ and $E$ yield an undirected graph $G(V, E)$. 
Formulating the Image Segmentation Problem

- Let $V$ be the set of pixels in an image.
- Let $E$ be the set of pairs of neighbouring pixels.
- $V$ and $E$ yield an undirected graph $G(V, E)$.
- Each pixel $i$ has a likelihood $a_i > 0$ that it belongs to the foreground and a likelihood $b_i > 0$ that it belongs to the background.
- These likelihoods are specified in the input to the problem.
Formulating the Image Segmentation Problem

- Let $V$ be the set of pixels in an image.
- Let $E$ be the set of pairs of neighbouring pixels.
- $V$ and $E$ yield an undirected graph $G(V, E)$.
- Each pixel $i$ has a likelihood $a_i > 0$ that it belongs to the foreground and a likelihood $b_i > 0$ that it belongs to the background.
- These likelihoods are specified in the input to the problem.
- We want the foreground/background boundary to be smooth:
Formulating the Image Segmentation Problem

- Let $V$ be the set of pixels in an image.
- Let $E$ be the set of pairs of neighbouring pixels.
- $V$ and $E$ yield an undirected graph $G(V, E)$.
- Each pixel $i$ has a likelihood $a_i > 0$ that it belongs to the foreground and a likelihood $b_i > 0$ that it belongs to the background.
- These likelihoods are specified in the input to the problem.
- We want the foreground/background boundary to be smooth: For each pair $(i, j)$ of pixels, assign separation penalty $p_{ij} \geq 0$ for placing one of them in the foreground and the other in the background.
The Image Segmentation Problem

Image Segmentation

INSTANCE: Pixel graphs $G(V, E)$, likelihood functions $a, b : V \rightarrow \mathbb{R}^+$, penalty function $p : E \rightarrow \mathbb{R}^+$

SOLUTION: Optimum labelling: partition of the pixels into two sets $A$ and $B$ that maximises

$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E \atop |A \cap \{i,j\}|=1} p_{ij}.$$
Developing an Algorithm for Image Segmentation

- There is a similarity between cuts and labellings.
- But there are differences:
  - We are maximising an objective function rather than minimising it.
  - There is no source or sink in the segmentation problem.
  - We have values on the nodes.
  - The graph is undirected.
Maximization to Minimization

Let \( Q = \sum_i (a_i + b_i) \).
Maximization to Minimization

- Let $Q = \sum_i (a_i + b_i)$.
- Notice that $\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i + \sum_{j \in B} a_j$.
- Therefore, maximising
  \[
  q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i, j) \in E \mid |A \cup \{i, j\}|=1} p_{ij}
  \]
  \[
  = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{(i, j) \in E \mid |A \cap \{i, j\}|=1} p_{ij}
  \]
  is identical to minimising
  \[
  q'(A, B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i, j) \in E \mid |A \cap \{i, j\}|=1} p_{ij}
  \]
Solving the Other Issues

- Solve the issues like we did earlier.

  ▶ Solve the issues like we did earlier.

  ▶ Add a new “super-source” \( s \) to represent the foreground.

  ▶ Add a new “super-sink” \( t \) to represent the background.

  ▶ Connect \( s \) and \( t \) to every pixel and assign capacity \( a_i \) to edge \((s, i)\) and capacity \( b_i \) to edge \((i, t)\).

  ▶ Direct edges away from \( s \) and into \( t \).

  ▶ Replace each edge in \( E \) with two directed edges of capacity 1.
Solving the Other Issues

- Solve the issues like we did earlier.
- Add a new “super-source” $s$ to represent the foreground.
- Add a new “super-sink” $t$ to represent the background.
Solving the Other Issues

- Solve the issues like we did earlier.
- Add a new “super-source” \( s \) to represent the foreground.
- Add a new “super-sink” \( t \) to represent the background.
- Connect \( s \) and \( t \) to every pixel and assign capacity \( a_i \) to edge \((s, i)\) and capacity \( b_i \) to edge \((i, t)\).
- Direct edges away from \( s \) and into \( t \).
- Replace each edge in \( E \) with two directed edges of capacity 1.
Cuts in the Flow Network

- Let $G'$ be this flow network and $(A, B)$ an $s$-$t$ cut.
- What does the capacity of the cut represent?
Cuts in the Flow Network

- Let $G'$ be this flow network and $(A, B)$ an $s$-$t$ cut.
- What does the capacity of the cut represent?
- Edges crossing the cut are of three types:

Figure 7.19 An $s$-$t$ cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.
Cuts in the Flow Network

Let $G'$ be this flow network and $(A, B)$ an $s$-$t$ cut.

What does the capacity of the cut represent?

Edges crossing the cut are of three types:

- $(s, w), w \in B$ contributes $a_w$.
- $(u, t), u \in A$ contributes $b_u$.
- $(u, w), u \in A, w \in B$ contributes $p_{uw}$.

Figure 7.19: An $s$-$t$ cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.
Cuts in the Flow Network

Let $G'$ be this flow network and $(A, B)$ an $s$-$t$ cut.

What does the capacity of the cut represent?

Edges crossing the cut are of three types:

- $(s, w), w \in B$ contributes $a_w$.
- $(u, t), u \in A$ contributes $b_u$.
- $(u, w), u \in A, w \in B$ contributes $p_{uw}$.

\[
c(A, B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i, j) \in E, |A \cap \{i, j\}|=1} p_{ij} = q'(A, B).
\]

Figure 7.19 An $s$-$t$ cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.
Solving the Image Segmentation Problem

- The capacity of a $s$-$t$ cut $c(A, B)$ exactly measures the quantity $q'(A, B)$.
- To maximise $q(A, B)$, we simply compute the $s$-$t$ cut $(A, B)$ of minimum capacity.
- Deleting $s$ and $t$ from the cut yields the desired segmentation of the image.