Greedy Algorithms

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Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
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- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.
Interval Scheduling

**INSTANCE:** Nonempty set \{ (s(i), f(i)), 1 \leq i \leq n \} of start and finish times of \( n \) jobs.

**SOLUTION:** The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
Template for Greedy Algorithm

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
- Key question: in what order should we process the jobs?
Template for Greedy Algorithm

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
- Key question: in what order should we process the jobs?
  - Earliest start time Increasing order of start time $s(i)$.
  - Earliest finish time Increasing order of finish time $f(i)$.
  - Shortest interval Increasing order of length $f(i) - s(i)$.
  - Fewest conflicts Increasing order of the number of conflicting jobs. How fast can you compute the number of conflicting jobs for each job?
Greedy Ideas that Do Not Work

Figure 4.1 Some instances of the Interval Scheduling Problem on which natural greedy algorithms fail to find the optimal solution. In (a), it does not work to select the interval that starts earliest; in (b), it does not work to select the shortest interval; and in (c), it does not work to select the interval with the fewest conflicts.
Interval Scheduling Algorithm: Earliest Finish Time

- Schedule jobs in order of earliest finish time (EFT).

Initially let $R$ be the set of all requests, and let $A$ be empty
While $R$ is not yet empty
  - Choose a request $i \in R$ that has the smallest finishing time
  - Add request $i$ to $A$
  - Delete all requests from $R$ that are not compatible with request $i$
EndWhile
Return the set $A$ as the set of accepted requests
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Return the set $A$ as the set of accepted requests

- Claim: $A$ is a compatible set of requests.
Analysing the EFT Algorithm

- Let $O$ be an optimal set of requests. We will show that $|A| = |O|$.
- Let $i_1, i_2, \ldots, i_k$ be the set of requests in $A$ in order.
- Let $j_1, j_2, \ldots, j_m$ be the set of requests in $O$ in order, $m \geq k$.
- Claim: For all indices $r \leq k$, $f(i_r) \leq f(j_r)$. 

Prove by induction on $r$. 

Claim: $m = k$. 

Claim: The greedy algorithm returns an optimal set $A$. 


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![Diagram](image)

**Figure 4.3** The inductive step in the proof that the greedy algorithm stays ahead.
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![Diagram: The inductive step in the proof that the greedy algorithm stays ahead.]

- Claim: $m = k$.
- Claim: The greedy algorithm returns an optimal set $A$. 

Can the greedy algorithm’s $r^{th}$ interval really finish later?
Implementing the EFT Algorithm

1. Reorder jobs so that they are in increasing order of finish time.
2. Store starting time of jobs in an array $S$.
3. Always select first interval. Let finish time be $f$.
4. Iterate over $S$ to find the first index $i$ such that $S[i] \geq f$. 

Running time is $O(n \log n)$, dominated by sorting.
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Interval Partitioning

**INSTANCE:** Set \{ (s(i), f(i)), 1 \leq i \leq n \} of start and finish times of \( n \) jobs.

**SOLUTION:** A partition of the jobs into \( k \) sets, where each set of jobs is mutually compatible, and \( k \) is minimised.

- This problem models the situation where you a set of fixed jobs, and you want to schedule all jobs using as few resources as possible.
The depth of a set of intervals is the maximum number of intervals that contain any time point.
Depth of Intervals

The depth of a set of intervals is the maximum number of intervals that contain any time point.

Claim: In any instance of **Interval Partitioning**, $k \geq \text{depth}$. 

Figure 4.4 (a) An instance of the Interval Partitioning Problem with ten intervals (a through j). (b) A solution in which all intervals are scheduled using three resources: each row represents a set of intervals that can all be scheduled on a single resource.
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Is it possible to compute $k$ efficiently? Is $k = \text{depth}$?

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Interval Partitioning Algorithm

- Compute the depth $d$ of the intervals.
**Interval Partitioning Algorithm**

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  ▶ Sort the intervals by their start times, breaking ties arbitrarily
  Let $I_1, I_2, \ldots, I_n$ denote the intervals in this order
  For $j = 1, 2, 3, \ldots, n$
    - For each interval $I_i$ that precedes $I_j$ in sorted order and overlaps it
      - Exclude the label of $I_i$ from consideration for $I_j$
    Endfor
  If there is any label from $\{1, 2, \ldots, d\}$ that has not been excluded then
    - Assign a nonexcluded label to $I_j$
  Else
    - Leave $I_j$ unlabeled
  Endif
Endfor

- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.
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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.
- Claim: The greedy algorithm is optimal.
- The running time of the algorithm is $O(n \log n)$. 
Scheduling to Minimise Lateness

- Study different model: job $i$ has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.
- A job $i$ is *delayed* if $f(i) > d(i)$; the *lateness of the job* is $\max(0, f(i) - d(i))$.
- The *lateness of a schedule* is $\max_i \left( \max (0, f(i) - d(i)) \right)$. 
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- Our goal is to assign a starting time \( s(i) \) to each job such that each job is delayed as little as possible.
- A job \( i \) is delayed if \( f(i) > d(i) \); the lateness of the job is \( \max(0, f(i) - d(i)) \).
- The lateness of a schedule is \( \max_i \left( \max \left( 0, f(i) - d(i) \right) \right) \).

Minimise Lateness

**INSTANCE:** Set \( \{(t(i), d(i)), 1 \leq i \leq n\} \) of lengths and deadlines of \( n \) jobs.

**SOLUTION:** Set \( \{s(i), 1 \leq i \leq n\} \) of start times such that \( \max_i \left( \max \left( 0, s(i) + t(i) - d(i) \right) \right) \) is as small as possible.
Template for Greedy Algorithm

- Key question: In what order should we schedule the jobs?
Template for Greedy Algorithm

Key question: In what order should we schedule the jobs?

- **Shortest length** Increasing order of length $t(i)$.
- **Shortest slack time** Increasing order of $d(i) - t(i)$.
- **Earliest deadline** Increasing order of deadline $d(i)$. 
**Algorithm for Minimising Lateness: Earliest Deadline First (EDF)**

Order the jobs in order of their deadlines
Assume for simplicity of notation that $d_1 \leq \ldots \leq d_n$
Initially, $f = s$
Consider the jobs $i = 1, \ldots, n$ in this order
  Assign job $i$ to the time interval from $s(i) = f$ to $f(i) = f + t_i$
  Let $f = f + t_i$
End
Return the set of scheduled intervals $[s(i), f(i)]$ for $i = 1, \ldots, n$

- Proof of correctness is more complex.
- We will use an exchange argument: gradually modify the optimal schedule $O$ till it is the same as the schedule $A$ computed by the algorithm.
Properties of Schedules

- A schedule has an *inversion* if a job $i$ with deadline $d(i)$ is scheduled before a job $j$ with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
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- Claim 1: The algorithm produces a schedule with no inversions and no idle time.

- Claim 2: All schedules with no inversions and no idle time have the same lateness.

- Claim 3: There is an optimal schedule with no idle time.

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Proving Claim 4

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Claim 4: There is an optimal schedule with no inversions and no idle time.

Approach: Start with an optimal schedule $O$ and use an *exchange argument* to convert $O$ into a schedule that satisfies Claim 4 and has lateness not larger than $O$. 

1. If $O$ has an inversion, then there is a pair of jobs $i$ and $j$ such that $d(j) < d(i)$.
2. Let $i$ and $j$ be consecutive inverted jobs in $O$. After swapping $i$ and $j$, we get a schedule $O'$ with one less inversion.
3. The maximum lateness of $O'$ is no larger than the maximum lateness of $O$.

It is enough to prove the last item, since after $(n^2)$ swaps, we obtain a schedule with no inversions whose maximum lateness is no larger than $O$. 
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Claim 4: There is an optimal schedule with no inversions and no idle time.

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- It is enough to prove the last item, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose maximum lateness is no larger than that of $O$. 
**Swapping Inverted Jobs**

In $O$, assume each request $r$ is scheduled for the interval $[s(r), f(r)]$ and has lateness $l(r)$. For $O'$, let the lateness of request $r$ be $l'(r)$.

![Diagram showing before and after swapping of inverted jobs](image)

**Claim:** $l'(k) = l(k)$, for all $k \neq i, j$.

**Claim:** $l'(j) \leq l(j)$.

**Claim:** $l'(i) \leq l(j)$ because $l'(i) = f(j) - d_i \leq f(j) - d_j = l(j)$.

![Figure 4.6](image) The effect of swapping two consecutive, inverted jobs.
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Figure 4.6 The effect of swapping two consecutive, inverted jobs.
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**Figure 4.6** The effect of swapping two consecutive, inverted jobs.

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<thead>
<tr>
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Only the finishing times of $i$ and $j$ are affected by the swap.
### Swapping Inverted Jobs

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The effect of swapping two consecutive, inverted jobs.

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Summary

- Greedy algorithms make local decisions.

- Three analysis strategies:
  
  **Greedy algorithm stays ahead**  Show that after each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.

  **Structural bound**  First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.

  **Exchange argument**  Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.