Analysis of Algorithms

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January 27, 2009
What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
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- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
- Goal: Develop algorithms that provably run quickly and use low amounts of space.
Worst-case Running Time

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- Bound the largest possible running time the algorithm over all inputs of size \( n \), as a function of \( n \).
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- Why worst-case? Why not average-case or on random inputs?
- Input size = number of elements in the input. Values in the input do not matter.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.
Polynomial Time

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  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.

Running time is $n^n$. Unacceptable in practice!

Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor $c$.

An algorithm has a polynomial running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size $n$, the running time of the algorithm is bounded by $cn^d$ steps.

Definition: An algorithm is efficient if it has a polynomial running time.
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Upper and Lower Bounds

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Asymptotic upper bound: A function \( f(n) \) is \( O(g(n)) \) if there exist constants \( c > 0 \) and \( n_0 \geq 0 \) such that for all \( n \geq n_0 \), we have \( f(n) \leq cg(n) \).

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Asymptotic lower bound: A function \( f(n) \) is \( \Omega(g(n)) \) if there exist constants \( c > 0 \) and \( n_0 \geq 0 \) such that for all \( n \geq n_0 \), we have \( f(n) \geq cg(n) \).

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- In these definitions, $c$ is a constant independent of $n$.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$. 
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Transitivity

- If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
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- If \( f = O(g) \), then \( f + g = \Theta(g) \).
Examples

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- For every $x > 0$, $\log n = O(n^x)$. 
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- For every $x > 0$, $\log n = O(n^x)$.
- For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$. 
Linear Time

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- Finding the minimum, merging two sorted lists.
- Sub-linear time. Binary search in a sorted array of \( n \) numbers takes \( O(\log n) \) time.
\( O(n \log n) \) **Time**

- Any algorithm where the costliest step is sorting.
Quadratic Time

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- Given a set of $n$ points in the plane, find the pair that are the closest.
Quadratic Time

- Enumerate all pairs of elements.
- Given a set of $n$ points in the plane, find the pair that are the closest. Surprising fact: can solve this problem in $O(n \log n)$ time later in the semester.
$O(n^k)$ Time

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Running time is $O(k^2 \binom{n}{k}) = O(n^k)$. 

$O(n^k)$ Time
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- Algorithm: For each \( 1 \leq i \leq n \), check if the graph has an independent size of size \( i \). Output largest independent set found.
- What is the running time?

\[ O\left(n^2 \log n\right) \]
Beyond Polynomial Time

- What is the largest size of an independent set in a graph with \( n \) nodes?
- Algorithm: For each \( 1 \leq i \leq n \), check if the graph has an independent size of size \( i \). Output largest independent set found.
- What is the running time? \( O(n^2 2^n) \).