Analysis of Algorithms

T. M. Murali

January 16, 2008
What is algorithm analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
What is algorithm analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
- Goal: Develop algorithms that provably run quickly and use low amounts of space.
Worst-case running time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$. 
Worst-case running time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$.
- Why worst-case? Why not average-case or on random inputs?
Worst-case running time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$.
- Why worst-case? Why not average-case or on random inputs?
- Input size = number of elements in the input.
Worst-case running time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$.
- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input. Values in the input do not matter.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.
Polynomial time

- Brute force algorithm: Check every possible solution.
Polynomial time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given $n$ numbers, permute them so that they appear in increasing order?
Polynomial time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given $n$ numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
Polynomial time

- Brute force algorithm: Check every possible solution.

- What is a brute force algorithm for sorting: given $n$ numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!
Polynomial time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given $n$ numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor $c$. 

...
Polynomial time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given \( n \) numbers, permute them so that they appear in increasing order?
  - Try all possible \( n! \) permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is \( nn! \). Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor \( c \).
- An algorithm has a \textit{polynomial} running time if there exist constants \( c > 0 \) and \( d > 0 \) such that on every input of size \( n \), the running time of the algorithm is bounded by \( cn^d \) steps.
Polynomial time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given \( n \) numbers, permute them so that they appear in increasing order?
  - Try all possible \( n! \) permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is \( nn! \). Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor \( c \).
- An algorithm has a \textit{polynomial} running time if there exist constants \( c > 0 \) and \( d > 0 \) such that on every input of size \( n \), the running time of the algorithm is bounded by \( cn^d \) steps.

Definition

An algorithm is \textit{efficient} if it has a polynomial running time.
Upper and lower bounds

Definition
Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq cg(n)$.

Definition
Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

Definition
Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. 
Upper and lower bounds

Definition
Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq cg(n)$.

Definition
Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

Definition
Asymptotic tight bound: A function $f(n)$ is \(\Theta(g(n))\) if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

- In these definitions, $c$ is a constant independent of $n$.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$. 
Properties of Asymptotic Growth Rates

Transitivity

- If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Theta(h) \), then \( f = \Theta(h) \).
Properties of Asymptotic Growth Rates

Transitivity

- If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Theta(h) \), then \( f = \Theta(h) \).

Additivity

- If \( f = O(h) \) and \( g = O(h) \), then \( f + g = O(h) \).
- Similar statements hold for lower and tight bounds.
Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity

- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.
- If $k$ is a constant and there are $k$ functions $f_i = O(h), 1 \leq i \leq k$, then $f_1 + f_2 + \ldots + f_k = O(h)$.
Properties of Asymptotic Growth Rates

Transitivity

- If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Theta(h) \), then \( f = \Theta(h) \).

Additivitvity

- If \( f = O(h) \) and \( g = O(h) \), then \( f + g = O(h) \).
- Similar statements hold for lower and tight bounds.
- If \( k \) is a constant and there are \( k \) functions \( f_i = O(h) \), \( 1 \leq i \leq k \), then \( f_1 + f_2 + \ldots + f_k = O(h) \).
Properties of Asymptotic Growth Rates

Transitivity

▶ If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
▶ If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
▶ If \( f = \Theta(g) \) and \( g = \Theta(h) \), then \( f = \Theta(h) \).

Additivity

▶ If \( f = O(h) \) and \( g = O(h) \), then \( f + g = O(h) \).
▶ Similar statements hold for lower and tight bounds.
▶ If \( k \) is a constant and there are \( k \) functions \( f_i = O(h), 1 \leq i \leq k \), then \( f_1 + f_2 + \ldots + f_k = O(h) \).
▶ If \( f = O(g) \), then \( f + g = \)
Properties of Asymptotic Growth Rates

Transitivity

▶ If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
▶ If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
▶ If \( f = \Theta(g) \) and \( g = \Theta(h) \), then \( f = \Theta(h) \).

Additivity

▶ If \( f = O(h) \) and \( g = O(h) \), then \( f + g = O(h) \).
▶ Similar statements hold for lower and tight bounds.
▶ If \( k \) is a constant and there are \( k \) functions \( f_i = O(h), 1 \leq i \leq k \), then \( f_1 + f_2 + \ldots + f_k = O(h) \).
▶ If \( f = O(g) \), then \( f + g = \Theta(g) \).
Examples

- $f(n) = pn^2 + qn + r$ is
Examples

- \( f(n) = pn^2 + qn + r \) is \( \theta(n^2) \). Can ignore lower order terms.
Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
Examples

- \( f(n) = pn^2 + qn + r \) is \( \theta(n^2) \). Can ignore lower order terms.
- Is \( f(n) = pn^2 + qn + r = O(n^3) \)?
- \( f(n) = \sum_{0 \leq i \leq d} a_i n^i = \)
Examples

- \( f(n) = pn^2 + qn + r \) is \( \theta(n^2) \). Can ignore lower order terms.
- Is \( f(n) = pn^2 + qn + r = O(n^3) \)?
- \( f(n) = \sum_{0 \leq i \leq d} a_in^i = O(n^d) \), if \( d > 0 \) is an integer constant and \( a_d > 0 \). Definition of \textit{polynomial time}
Examples

- \( f(n) = pn^2 + qn + r \) is \( \theta(n^2) \). Can ignore lower order terms.
- Is \( f(n) = pn^2 + qn + r = O(n^3) \)?
- \( f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d) \), if \( d > 0 \) is an integer constant and \( a_d > 0 \). Definition of polynomial time
- Is an algorithm with running time \( O(n^{1.59}) \) a polynomial-time algorithm?
Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$. Definition of *polynomial time*
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every $x > 0$, $\log n = O(n^x)$.
Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$. Definition of \textit{polynomial time}
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every $x > 0$, $\log n = O(n^x)$.
- For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$. 
Linear time

- Running time is at most a constant factor times the size of the input.
**Linear time**

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
### Linear time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Sub-linear time.
Linear time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Sub-linear time. Binary search in a sorted array of $n$ numbers takes $O(\log n)$ time.
$O(n \log n)$ time

- Any algorithm where the costliest step is sorting.
**Quadratic time**

- Enumerate all pairs of elements.
Quadratic time

- Enumerate all pairs of elements.
- Given a set of $n$ points in the plane, find the pair that are the closest.
Quadratic time

- Enumerate all pairs of elements.
- Given a set of $n$ points in the plane, find the pair that are the closest. Surprising fact: can solve this problem in $O(n \log n)$ time later in the semester.
$O(n^k)$ time

- Does a graph have an independent set of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that no two are joined by an edge?
Does a graph have an independent set of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that no two are joined by an edge?

Algorithm: For each subset $S$ of $k$ nodes, check if $S$ is an independent set. If the answer is yes, report it.

$O(n^k)$ time
$O(n^k)$ time

▶ Does a graph have an independent set of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that no two are joined by an edge?

▶ Algorithm: For each subset $S$ of $k$ nodes, check if $S$ is an independent set. If the answer is yes, report it.

▶ Running time is
Does a graph have an independent set of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that no two are joined by an edge?

Algorithm: For each subset $S$ of $k$ nodes, check if $S$ is an independent set. If the answer is yes, report it.

Running time is $O(k^2 \binom{n}{k}) = O(n^k)$. 

$O(n^k)$ time
Beyond polynomial time

- What is the largest size of an independent set in a graph with $n$ nodes?
Beyond polynomial time

- What is the largest size of an independent set in a graph with \( n \) nodes?
- Algorithm: For each \( 1 \leq i \leq n \), check if the graph has an independent size of size \( i \). Output largest independent set found.
Beyond polynomial time

▶ What is the largest size of an independent set in a graph with $n$ nodes?
▶ Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size $i$. Output largest independent set found.
▶ What is the running time?

$O(n^2)$. 

T. M. Murali
January 16, 2008
Analysis of Algorithms
What is the largest size of an independent set in a graph with $n$ nodes?

Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size $i$. Output largest independent set found.

What is the running time? $O(n^22^n)$.