CS 4604: Introduction to Database Management Systems

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Lecture #14: Functional Dependencies
Example

- Convert to relations
  - Students (ID, Name)
  - Advisors (ID, Name)
  - Favourite (StudentID, AdvisorID)
  - Advises (StudentID, AdvisorID)
Example

- What if we combine Students, Advises, and Favourite into one relation?
  - Students(\text{Id}, \text{Name}, \text{AdvisorId}, \text{AdvisorName}, \text{FavouriteAdvisorId})
  - Seems ‘intuitively bad’ right?
Example of a Bad Relation

Students(Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)

- What makes it bad?
- Given the Student’s Id, can any other values be determined?
  - Name and FavouriteAdvisorId
  - Id \(\rightarrow\) Name
  - Id \(\rightarrow\) FavouriteAdvisorId
Example of a Bad Relation

Students(Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)

- Name and FavouriteAdvisorId
- Id $\rightarrow$ Name
- Id $\rightarrow$ FavouriteAdvisorId
- AdvisorId $\rightarrow$ ?
Example of a Bad Relation

Students(Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)

- Name and FavouriteAdvisorId
- Id → Name
- Id → FavouriteAdvisorId
- AdvisorId → AdvisorName

- Can we say Id → AdvisorId?
  - Not really! Why?
  - Id is not a key for Students relation
  - Key: {Id, AdvisorId}
Example of a Bad Relation

Students(Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)

- OK, what **really** makes it bad?
- Ans: Parts of the key determine other attributes
- Leads to:
  - Redundancy (Space, Inconsistencies, ....)
Motivation for Functional Dependencies

- Reason about constraints on attributes in a relation
- Procedurally determine the keys of a relation
- Detect when a relation has redundant information
- Improve database designs systematically using normalization
Definition of FD (Functional Dependency)

- X \rightarrow Y

‘X’ functionally determines ‘Y’
Informally: ‘if you know ‘X’, there is only one ‘Y’ to match’
Definition of FD (Functional Dependency)

- ((If \( t \) is a tuple in a relation \( R \) and \( A \) is an attribute of \( R \), then \( t[A] \) is the value of attribute \( A \) in tuple \( t \))

- Formally:
  \[ X \rightarrow Y \Rightarrow (t1[X] = t2[X] \Rightarrow t1[Y] = t2[Y]) \]

if two tuples agree on the ‘\( X \)’ attribute, they *must* agree on the ‘\( Y \)’ attribute, too

(eg., if ids are the same, so should be names)
X \rightarrow Y

- X and Y can be sets of attributes
- Definition of FDs
- A FD on a relation R is a statement:
  - If two tuples in R agree on attributes A1, A2, ..., An they agree on attribute B
  - Notation: A1 A2 ... An \rightarrow B
A FD is a constraint on a single relational schema

— It must hold on every instance of the relation

— You can not deduce an FD from a relation instance!
Examples of FDs

- List the FDs

Courses(Number, DeptName, CourseName, Classroom, Enrollment)

<table>
<thead>
<tr>
<th>Number</th>
<th>DeptName</th>
<th>CourseName</th>
<th>Classroom</th>
<th>Enrollment</th>
</tr>
</thead>
<tbody>
<tr>
<td>4604</td>
<td>CS</td>
<td>Databases</td>
<td>TORG 1020</td>
<td>45</td>
</tr>
<tr>
<td>4604</td>
<td>Dance</td>
<td>Tree Dancing</td>
<td>Drillfield</td>
<td>45</td>
</tr>
<tr>
<td>4604</td>
<td>English</td>
<td>The Basis of Data</td>
<td>Williams 44</td>
<td>45</td>
</tr>
<tr>
<td>2604</td>
<td>CS</td>
<td>Data Structures</td>
<td>MCB 114</td>
<td>100</td>
</tr>
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<td>Physics</td>
<td>Dark Matter</td>
<td>Williams 44</td>
<td>100</td>
</tr>
</tbody>
</table>

- Number DeptName → CourseName
- Number DeptName → Classroom
- Number DeptName → Enrollment
- Number DeptName → CourseName Classroom Enrollment
Examples of FDs

- List the FDs

Courses(Number, DeptName, CourseName, Classroom, Enrollment)

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- Is Number → Enrollment an FD?
Where do FDs come from?

- “Keyness” of attributes
- Domain and application constraints
- Real world constraints
  - E.g. ProfessorID Time → Classroom
Definition of Keys

- FDs allow us to formally define keys
- A set of attributes \{A_1, A_2, ..., A_n\} is a key for relation R if:

**Uniqueness:** \{A_1, A_2, ..., A_n\} functionally determine all the other attributes of R

**Minimality:** no proper set of \{A_1, A_2, ..., A_n\} functionally determines all other attributes of R.
Definitions of Keys

- A superkey is a set of attributes that has the uniqueness property but is not necessarily minimal.
- If a relation has multiple keys, specify one to be primary key.
- Convention: underline the attributes (but you know that!)
- If a key has only one attribute $A$, say $A$ rather than $\{A\}$.
Example of keys

- What is the key for Courses (Number, DeptName, CourseName, Classroom, Enrollment) ?

- The key is \{Number, DeptName\}
  - These attributes functionally determine every other attribute
  - No proper subset of \{Number, DeptName\} has this property
Example of Keys

- What is the key for Teach (Number, DepartmentName, ProfessorName, Classroom) ?

- The key is \{Number, DepartmentName\}  
  – Why?
Keys in E/R to Relational Conversion

- From an ENTITY SET

If the relation comes from an entity set, the key attributes of the relation are precisely the key attributes of the entity set
Keys in E/R to Relational Conversion

- From a RELATIONSHIP (binary for now between E and F)
  - R is many-many:
    - Key attributes of the relation are the key attributes of E and of F
  - R is many-one:
    - Key attributes of the relation are the key attributes of E
  - R is one-one:
    - Key attributes of the relation are the key attributes of E or of F
Keys in E/R to Relational Conversion

- From a RELATIONSHIP (multiway?)
- Need to reason about the FDs that R satisfies
- No simple rule
- If R has an arrow towards entity set E, at least one key for the relation for R excludes the key for E
Rules for Manipulating FDs

- Learn how to reason about FDs
- Define rules for deriving new FDs from a given set of FDs
- Example: R (A, B, C) satisfies FDs $A \rightarrow B$, $B \rightarrow C$.
  - What others does it satisfy?
  - $A \rightarrow C$
  - What is the key for R?
  - A (as $A \rightarrow B$ and $A \rightarrow C$)
Equivalence of FDs

- Why?
  - To derive new FDs from a set of FDs
- An FD F follows from a set of FDs T if every relation instance that satisfies all the FDs in T also satisfies F
  - $A \rightarrow C$ follows from $T = \{A \rightarrow B, B \rightarrow C\}$
- Two sets of FDs S and T are equivalent if each FD in S follows from T and each FD in T follows from S
  - $S = \{A \rightarrow B, B \rightarrow C, A \rightarrow C\}$ and $T = \{A \rightarrow B, B \rightarrow C\}$ are equivalent
Splitting and Combining FDs

- The set of FDs
  - A1 A2 A3...An → B1
  - A1 A2 A3...An → B2
  - ...
  is equivalent to the FD
  - A1 A2 A3...An → B1 B2 B3 ... Bm

- This equivalence implies two rules:
  - Splitting rule
  - Combining rule
  - These rules work because all the FDs in S and T have identical left hand sides
Splitting and Combining FDs

- Can we split and combine left hand sides of FDs?
- For the relation Courses, is the FD
  - Number DeptName $\rightarrow$ CourseName
equivalent to the set of FDs
  - \{Number $\rightarrow$ CourseName, DeptName $\rightarrow$ CourseName\}?
  - NO
Triviality of FDs

- A FD $A_1 A_2 \ldots A_n \rightarrow B_1 B_2 \ldots B_m$ is
  - Trivial if the $B$'s are a subset of the $A$’s
    \[
    \{B_1, B_2, \ldots B_n\} \subseteq \{A_1, A_2, \ldots A_n\}
    \]
  - Non-trivial if at least one $B$ is not among the $A$’s
    \[
    \{B_1, B_2, \ldots B_n\} - \{A_1, A_2, \ldots A_n\} \neq \emptyset
    \]
  - Completely non-trivial if none of the $B$’s are among the $A$’s
    \[
    \{B_1, B_2, \ldots B_n\} \cap \{A_1, A_2, \ldots A_n\} = \emptyset
    \]
Triviality of FDs

- What good are trivial and non-trivial FDs?
  - Trivial dependencies are always true
  - They help simplify reasoning about FDs

- Trivial dependency rule: The FD $A_1\, A_2\ldots\, A_n \rightarrow B_1\, B_2\ldots\, B_m$ is equivalent to the FD $A_1\, A_2\ldots\, A_n \rightarrow C_1\, C_2\ldots\, C_k$, where the $C$’s are those $B$’s that are not $A$’s i.e.

$$\{C_1,\, C_2,\ldots,\, C_k\} = \{B_1,\, B_2,\ldots,\, B_m\} - \{A_1,\, A_2,\ldots,\, A_n\}$$
Suppose a relation $R$ (A, B, C, D, E, F) has FDs:
- $AB \rightarrow C$, $BC \rightarrow AD$, $D \rightarrow E$, $CF \rightarrow B$

**Question:**
Find set $X$ of attributes such that $AB \rightarrow X$ is true

**Answer:**
$X = \{A, B, C, D, E\}$ i.e. $AB \rightarrow ABCDE$
Suppose a relation R (A, B, C, D, E, F) has FDs:
- AB \rightarrow C, BC \rightarrow AD, D \rightarrow E, CF \rightarrow B

Question:
Find set Y of attributes such that BCF \rightarrow Y is true

Answer:
Y = \{A, B, C, D, E, F\} i.e. BCF \rightarrow ABCDEF
Closure of Attributes: Example

- Suppose a relation R (A, B, C, D, E, F) has FDs:
  - AB → C, BC → AD, D → E, CF → B

- Question:
  Find set Z of attributes such that AF → Z is true

- Answer:
  Y = {A, F} i.e. AF → AF
Closure of Attributes: Example

- Suppose a relation $R (A, B, C, D, E, F)$ has FDs:
  - $AB \rightarrow C$, $BC \rightarrow AD$, $D \rightarrow E$, $CF \rightarrow B$

- $X$, $Y$, $Z$ are the closures of $\{A, B\}$, $\{B, C, F\}$, and $\{A, F\}$ respectively
Closure of Attributes: Definition

- **Given:**
  - Attributes \{A_1, A_2, \ldots, A_n\}
  - A set of FDs \(S\)

- **The Closure** of \{A_1, A_2, \ldots, A_n\} under \(S\) is
  - the set of attributes \{B_1, B_2, \ldots, B_m\} such that for \(1 \leq i \leq m\), the FD \(A_1 A_2 \ldots A_n \rightarrow B_i\) follows from \(S\)
  - the closure is denoted by \(\{A_1, A_2, \ldots, A_n\}^+\)
Closure of Attributes: Definition

- Question:
Which attributes must \{A_1, A_2, \ldots, A_n\}^+ contain at the minimum?

- Answer:
\{A_1, A_2, \ldots, A_n\}

- Why?
A_1 A_2 \ldots A_n \rightarrow A_i \text{ is a trivial FD}
Closure of Attributes: Algorithm

- **Given (INPUT):**
  - Attributes \{A_1, A_2, \ldots, A_n\}
  - Set of FDs \(S\)

- **Find (OUTPUT):**
  - \(X = \{A_1, A_2, \ldots, A_n\}^+\)
Closure of Attributes: Algorithm

1. Use the splitting rule so that each FD in S has one attribute on the right.
2. Set \( X = \{A_1, A_2 ..., A_n\} \)
3. Find FD \( B_1 B_2...B_k \rightarrow C \) in S such that \( \{B_1 B_2 ... B_k\} \subseteq X \) but \( C \not\in X \)
4. Add C to X
5. Repeat the last two steps until you can’t find C

Why is the algorithm correct? Read 3.2.5 in textbook
Why compute Closures?

- Prove correctness of rules for manipulating FDs

Example:
Prove the transitive rule i.e.

IF
A1 A2 ... An \(\rightarrow\) B1 B2 ... Bm
B1 B2 ... Bm \(\rightarrow\) C1 C2 ... Ck
THEN
A1 A2 ... An \(\rightarrow\) C1 C2 ... Ck

To prove this, check if
\[
\{c_1, c_2, \ldots, c_k\} \subseteq \{a_1, a_2, \ldots, a_n\}^+ 
\]
Why compute closures?

- Check if a “new” FD $A_1, A_2, \ldots, A_n \rightarrow B$ follows from a set of FDs $S$
  
  Simply check if $B$ is in $\{A_1, A_2, \ldots, A_n\}^+$ under $S$

- Get keys procedurally (aka algorithmically)

A set of attributes $X$ is a key for a relation $R$ iff

- $\{X\}^+$ is the set of all attributes of $R$
- For no attribute $A \in X$ is $\{X - \{A\}\}^+$ the set of all attributes of $R$
Examples of Closure Computations

- Consider the “bad” relation Students (Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)
- What are the FDs that hold in this relation?
  - Id → Name
  - Id → FavouriteAdvisorId
  - AdvisorId → AdvisorName
- To compute the key for this relation:
  - Compute the closures for all sets of attributes
  - Find the minimal set of attributes whose closure is the set of all attributes
Algorithm for computing keys

- **Given (INPUT)**:
  - A relation R (A1, A2, ..., An)
  - The set of all FDs S that hold in R

- **Find (OUTPUT)**:
  - Compute all the keys of R

1. For every subset K of {A1, A2, ..., An} compute its closure
2. If \( K^+ = \{A1, A2, ... An\} \) and for every attribute A, \( K - \{A\}^+ \) is not \( \{A1, A2, ... An\} \), then output K as a key

- Running time?
Armstrong’s Axioms

- We can use closures of attributes to determine if any FD follows from a given set of FDs.
- Armstrong's axioms: complete set of inference rules from which it is possible to derive every FD that follows from a given set:
Armstrong’s Axioms

- **Reflexivity**
  \[ Y \subseteq X \Rightarrow X \rightarrow Y \]
  – E.g. ssn, name \( \rightarrow \) ssn

- **Augmentation**
  \[ X \rightarrow Y \Rightarrow XW \rightarrow YW \]
  – E.g. ssn \( \rightarrow \) name then ssn grade \( \rightarrow \) name grade
Armstrong's Axioms

- **Transitivity**

\[
\begin{align*}
X \rightarrow Y \\
Y \rightarrow Z
\end{align*}
\] \Rightarrow X \rightarrow Z

e.g. if ssn \rightarrow address and address \rightarrow tax-rate
then

ssn \rightarrow address
Armstrong Axioms

- Sound and complete
- Additional rules
  - Union \( X \rightarrow Y \) \( X \rightarrow Z \) \Rightarrow X \rightarrow YZ
  - Decomposition \( X \rightarrow YZ \) \Rightarrow X \rightarrow Y \( X \rightarrow Z \) \( X \rightarrow \)
  - Pseudo-transitivity \( X \rightarrow Y \) \( YW \rightarrow Z \) \Rightarrow XW \rightarrow Z
Armstrong’s Axioms

- Prove ‘Union’ from three axioms:

\[
X \rightarrow Y \quad X \rightarrow Z \quad \Rightarrow \quad X \rightarrow YZ
\]
Armstrong’s Axioms

- Prove ‘Union’ from three axioms:

\[ X \rightarrow Y \quad (1) \]
\[ X \rightarrow Z \quad (2) \]
\[ (1) + \text{augm.} w / Z \Rightarrow XZ \rightarrow YZ \quad (3) \]
\[ (2) + \text{augm.} w / X \Rightarrow XX \rightarrow XZ \quad (4) \]

but \quad \text{XX is X thus}

(3) + (4) and transitivity \quad \Rightarrow X \rightarrow YZ
Armstrong’s Axioms

- Prove Pseudo-transitivity:

\[
\begin{align*}
Y \subseteq X & \implies X \rightarrow Y \\
X \rightarrow Y & \implies XW \rightarrow YW \\
X \rightarrow Y \quad Y \rightarrow Z & \implies X \rightarrow Z
\end{align*}
\]

\[
\begin{align*}
X \rightarrow Y \\
YW \rightarrow Z
\end{align*}
\]

\[
\implies XW \rightarrow Z
\]
Armstrong’s Axioms

- Prove Pseudo-transitivity:

\[
Y \subseteq X \Rightarrow X \rightarrow Y
\]
\[
X \rightarrow Y \Rightarrow XW \rightarrow YW
\]
\[
\begin{align*}
X \rightarrow Y \\
Y \rightarrow Z
\end{align*}
\] 
\[
\Rightarrow X \rightarrow Z
\]
\[
X \rightarrow YZ \Rightarrow \begin{cases} 
X \rightarrow Y \\
X \rightarrow Z
\end{cases}
\]
Note on notation

- Set of attributes: \{A1, A2, A3\}: curly braces surround attributes, attributes separated by commas
- FD: A1 A2 → A3: no parentheses or curly braces, attributes separated by spaces, arrows separates left hand side and right hand side
- Set of FDs: \{A1 A2 → A3, A2 → A1\}: curly braces surround FDs, FDs separated by commas
Computing Closures of FDs

- To compute the closure of a set of FDs, repeatedly apply Armstrong’s Axioms until you cannot find any new FDs.
Examples of Computing Closures of FDs

- (Let us include only completely non-trivial FDs in these examples, with a single attribute on the right))
- $F = \{A \rightarrow B, \ B \rightarrow C\}$
- $\{F\}^+ = ??$
Examples of Computing Closures of FDs

- ((Let us include only completely non-trivial FDs in these examples, with a single attribute on the right))
- \( F = \{A \rightarrow B, B \rightarrow C\} \)
- \( \{F\}^+ = \{A \rightarrow B, B \rightarrow C, A \rightarrow C, AC \rightarrow B, AB \rightarrow C\} \)
Examples of Computing Closures of FDs

- (Let us include only completely non-trivial FDs in these examples, with a single attribute on the right)

- $F = \{AB \rightarrow C, BC \rightarrow A, AC \rightarrow B\}$

- $\{F\}^+ = ??$
Examples of Computing Closures of FDs

- ((Let us include only completely non-trivial FDs in these examples, with a single attribute on the right))

- $F = \{AB \rightarrow C, BC \rightarrow A, AC \rightarrow B\}$

- $\{F\}+ = \{AB \rightarrow C, BC \rightarrow A, AC \rightarrow B\}$
Examples of Computing Closures of FDs

- ((Let us include only completely non-trivial FDs in these examples, with a single attribute on the right))
- $F = \{A \rightarrow B, B \rightarrow C, C \rightarrow D\}$
- $\{F\}^+ = ??$
Examples of Computing Closures of FDs

- Let us include only completely non-trivial FDs in these examples, with a single attribute on the right)
- \( F = \{A \rightarrow B, B \rightarrow C, C \rightarrow D\} \)
- \( \{F\}^+ = \{A \rightarrow B, B \rightarrow C, C \rightarrow D, A \rightarrow C, A \rightarrow D, B \rightarrow D, \ldots\} \)
Closures of Attributes vs Closure of FDs

- Both algorithms take as input a relation R and a set of FDs F

- Closure of FDs:
  - Computes \( \{F\}^+ \), the set of all FDs that follow from F
  - Output is a set of FDs
  - Output may contain an exponential number of FDs

- Closure of attributes:
  - In addition, takes a set \( \{A_1, A_2, ..., A_n\} \) of attributes as input
  - Computes \( \{A_1, A_2, ..., A_n\}^+ \), the set of all attributes \( B \), such that \( A_1 A_2 ... A_n \rightarrow B \) follows from F
  - Output is set of all attributes
  - Output may contain at most the number of attributes in R
A relation may have a large set of equivalent sets of FDs

If we are given a set $S$ of FDs, then any set of FDs that is equivalent to $S$ is called a basis of $S$

We will limit ourselves to bases in which each FD has only one attribute on the right hand side
Minimal Basis

- A set $B$ of FDs is a minimal basis for a relation $R$ if
  - Every FD in $B$ has one attribute on the right hand side
  - If we remove any FD from $B$, then the result is not a basis
  - If for any FD in $B$, we remove one or more attributes from the left hand side of the FD, then the result is not a basis
Example of Minimal Basis

- $R(A, B, C)$ is a relation such that each attribute functionally determines the other two attributes
- What are the FDs that hold in $R$ and what are the minimal bases?
  - (Assume only one attribute on the right-hand side, only non-trivial FDs)
Example of Minimal Basis

- R(A, B, C) is a relation such that each attribute functionally determines the other two attributes
- What are the FDs that hold in R and what are the minimal bases?
  - (Assume only one attribute on the right-hand side, only non-trivial FDs)
- FDs: A→B, A→C, B→A, B→C, C→A, C→B,
R(A, B, C) is a relation such that each attribute functionally determines the other two attributes

What are the FDs that hold in R and what are the minimal bases?

- (Assume only one attribute on the right-hand side, only non-trivial FDs)

FDs: A \rightarrow B, A \rightarrow C, B \rightarrow A, B \rightarrow C, C \rightarrow A, C \rightarrow B, AB \rightarrow C, BC \rightarrow A, AC \rightarrow B
Example of Minimal Basis

- R(A, B, C) is a relation such that each attribute functionally determines the other two attributes.
- What are the FDs that hold in R and what are the minimal bases?
  - (Assume only one attribute on the right-hand side, only non-trivial FDs)
- FDs: A \rightarrow B, A \rightarrow C, B \rightarrow A, B \rightarrow C, C \rightarrow A, C \rightarrow B, AB \rightarrow C, BC \rightarrow A, AC \rightarrow B
- Minimal Bases: \{A \rightarrow B, B \rightarrow A, B \rightarrow C, C \rightarrow B\}, \{A \rightarrow B, B \rightarrow C, C \rightarrow A\}, etc.