Manipulating Functional Dependencies

Zaki Malik

September 30, 2008

Definition of Functional Dependency

- If *t* is a tuple in a relation *R* and *A* is an attribute of *R*, then t_A is the value of attribute *A* in tuple *t*.

- The FD AdvisorId → AdvisorName holds in *R* if in every instance of *R*, for every pair of tuples *t* and *u*

$$\boxed{\text{if } t_{\texttt{AdvisorId}} = u_{\texttt{AdvisorId}}, \text{ then } t_{\texttt{AdvisorName}} = u_{\texttt{AdvisorName}}}$$

Rules for Manipulating FDs

- Learn how to reason about FDs.
- Define rules for deriving new FDs from a given set of FDs.
- Use these rules to remove "anomalies" from relational designs.
- Example: A relation R with attributes A, B, and C, satisfies the FDs A → B and B → C. What other FDs does it satisfy?

  A → C

- What is the key for R ?
  - A, because A → B and A → C

Splitting and Combining FDs

- Can we split and combine left hand sides of FDs?

  - For the relation `Courses` is the FD
    `Number DeptName` $\rightarrow$ `CourseName`
    equivalent to the set of FDs
    {`Number` $\rightarrow$ `CourseName`, `DeptName` $\rightarrow$ `CourseName`}?

  - **No !**

Triviality of FDs

An FD $A_1A_2\ldots A_n \rightarrow B_1B_2\ldots B_m$ is

- *trivial* if the B's are a subset of the A's,
  $\{B_1, B_2, \ldots B_n\} \subseteq \{A_1, A_2, \ldots A_n\}$
- *non-trivial* if at least one B is not among the A's,
  $\{B_1, B_2, \ldots B_n\} - \{A_1, A_2, \ldots A_n\} \neq \emptyset$
- *completely non-trivial* if none of the B's are among the A's, i.e.,
  $\{B_1, B_2, \ldots B_n\} \cap \{A_1, A_2, \ldots A_n\} = \emptyset$.
- *Trivial dependency rule*: The FD $A_1A_2\ldots A_n \rightarrow B_1B_2\ldots B_m$ is equivalent to the FD $A_1A_2\ldots A_n \rightarrow C_1C_2\ldots C_k$, where the C's are those B's that are not A's, i.e.,
  $\{C_1, C_2, \ldots, C_k\} = \{B_1, B_2, \ldots, B_m\} - \{A_1, A_2, \ldots, A_n\}$.
- What good are trivial and non-trivial dependencies?
  - Trivial dependencies are always true.
  - They help simplify reasoning about FDs.

Closure of FD sets

- Given a relation schema R and set S of FDs
  - is the FD *F* logically implied by S?
- Example
  - R = {A,B,C,G,H,I}
  - S = A → B, A → C, CG → H, CG → I, B → H
  - would A → H be logically implied?
  - yes (you can prove this, using the definition of FD)
- Closure of S: S^+ = all FDs logically implied by S
- How to compute S^+ ?
  - we can use <u>Armstrong's axioms</u>

Armstrong's Axioms

- Reflexivity rule
  - $A_1\ A_2 \ldots A_n \rightarrow$ *a subset of* $A_1\ A_2 \ldots A_n$
- Augmentation rule
  - $A_1\ A_2 \ldots A_n \rightarrow B_1\ B_2 \ldots B_m$

    then

    $$A_1\ A_2 \ldots A_n\ C_1\ C_2 \ldots C_k \rightarrow B_1\ B_2 \ldots B_m\ C_1\ C_2 \ldots C_k$$
- Transitivity rule
  - $A_1\ A_2 \ldots A_n \rightarrow B_1\ B_2 \ldots B_m$ and

    $B_1\ B_2 \ldots B_m \rightarrow C_1\ C_2 \ldots C_k$

    then

    $$A_1\ A_2 \ldots A_n \rightarrow C_1\ C_2 \ldots C_k$$

Inferring S^+ using Armstrong's Axioms

- S^+ = S
- Loop
  - For each *F* in S, apply reflexivity and augmentation rules
  - add the new FDs to S^+
  - For each pair of FDs in S, apply the transitivity rule
  - add the new FD to S^+
- Until S^+ does not change any further

Additional Rules

- Union rule
  - X → Y and X → Z, then X → YZ
  - (X, Y, Z are sets of attributes)

- Decomposition rule
  - X → YZ, then X → Y and X → Z

- Pseudo-transitivity rule
  - X → Y and YZ → U, then XZ → U

- These rules can be inferred from Armstrong's axioms

Example

- $R = (A, B, C, G, H, I)$
  $F = \{ A \rightarrow B \quad A \rightarrow C \quad CG \rightarrow H \quad CG \rightarrow I \quad B \rightarrow H\}$
- some members of F^+
  - $A \rightarrow H$
    - by transitivity from $A \rightarrow B$ *and* $B \rightarrow H$
  - $AG \rightarrow I$
    - by augmenting $A \rightarrow C$ with G, to get $AG \rightarrow CG$
      and then transitivity with $CG \rightarrow I$
  - $CG \rightarrow HI$
    - from $CG \rightarrow H$ *and* $CG \rightarrow I$: "union rule" can be inferred from
      - definition of functional dependencies, or
      - Augmentation of $CG \rightarrow I$ to infer $CG \rightarrow CGI$, augmentation of $CG \rightarrow H$ to infer $CGI \rightarrow HI$, and then transitivity

Closures of Attributes

Suppose a relation with attributes A, B, C, D, E, and F satisfies the FDs

$$AB \rightarrow C \quad BC \rightarrow AD \quad D \rightarrow E, \quad CF \rightarrow B$$

Given these FDs,

- what is the set X of attributes such that $AB \rightarrow X$ is true?
  $X = \{A, B, C, D, E\}$, i.e., $AB \rightarrow ABCDE$.

- what is the set Y of attributes such that $BCF \rightarrow Y$ is true?
  $Y = \{A, B, C, D, E, F\}$, i.e., $BCF \rightarrow ABCDEF$

- $\{B, C, F\}$ is a superkey.

Closures of Attributes: Definition

Given

- a set of attributes $\{A_1, A_2, \ldots, A_n\}$ and
- a set of FDs S,

the *closure* of $\{A_1, A_2, \ldots, A_n\}$ under the FDs in S is

- the set of attributes $\{B_1, B_2, \ldots, B_m\}$ such that for $1 \leq i \leq m$, the FD $A_1 A_2 \ldots A_n \rightarrow B_i$ follows from S.
- the closure is denoted by $\{A_1, A_2, \ldots, A_n\}^+$.

- Which attributes must $\{A_1, A_2, \ldots, A_n\}^+$ contain at a minimum? $\{A_1, A_2, \ldots, A_n\}$. Why?

  $A_1 A_2 \ldots A_n \rightarrow A_i$ is a trivial FD.

Closures of Attributes: Algorithm

Given

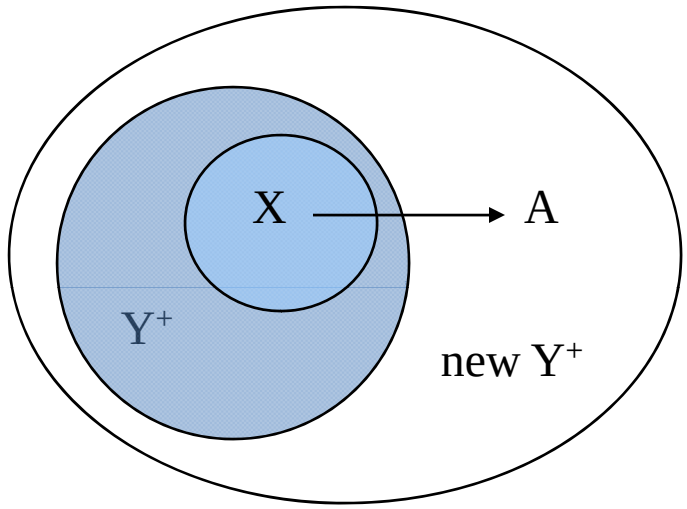
- a set of attributes $\{A_1, A_2, \ldots, A_n\}$ and
- a set of FDs S,
- compute $X = \{A_1, A_2, \ldots, A_n\}^+$.

1. Set $X \leftarrow \{A_1, A_2, \ldots, A_n\}$.
2. Find an FD $B_1 B_2 \ldots B_k \rightarrow C$ in S such that $\{B_1, B_2, \ldots B_k\} \subseteq X$ but $C \notin X$.
3. Add C to X.
4. Repeat the last two steps until you cannot find such an attribute C.
5. The final value of X is the desired closure.

Closures of Attributes: Algorithm

- Basis: $Y^+ = Y$
- Induction: Look for an FD's left side X that is a subset of the current Y^+
  - If the FD is $X \rightarrow A$, add A to Y^+

Diagramatically:



Why is the Concept of Closures Useful?

- Closures allow us to prove correctness of rules for manipulating FDs.
  - *Transitive rule*: if
    $A_1 A_2 \ldots A_n \rightarrow B_1 B_2 \ldots B_m$
    and
    $B_1 B_2 \ldots B_m \rightarrow C_1 C_2 \ldots C_n$
    then
    $A_1 A_2 \ldots A_n \rightarrow C_1 C_2 \ldots C_n$.
  - To prove this rule, simply check if
    $\{C_1, C_2, \ldots, C_n\} \subseteq \{A_1, A_2, \ldots, A_n\}^+$.
- Closures allow us to procedurally define keys. A set of attributes X is a key for a relation R if and only if
  - $\{X\}^+$ is the set of all attributes of R and
  - for no attribute $A \in X$ is $\{X - \{A\}\}^+$ the set of all attributes of R.

Uses of Attribute Closure

There are several uses of the attribute closure algorithm:

- Testing for superkey:
  - To test if α is a superkey, we compute $\alpha^{+,}$ and check if α^{+} contains all attributes of *R*.
- Testing functional dependencies
  - To check if a functional dependency $\alpha \rightarrow \beta$ holds (or, in other words, is in F^{+}), just check if $\beta \subseteq \alpha^{+}$.
  - That is, we compute α^{+} by using attribute closure, and then check if it contains β.
  - Is a simple and cheap test, and very useful
- Computing closure of F
  - For each $\gamma \subseteq R$, we find the closure γ^{+}, and for each $S \subseteq \gamma^{+}$, we output a functional dependency $\gamma \rightarrow S$.

Example of Attribute Set Closure

- $R = (A, B, C, G, H, I)$
- $F = \{A \rightarrow B \quad A \rightarrow C \quad CG \rightarrow H \quad CG \rightarrow I \quad B \rightarrow H\}$
- $(AG)^+$
  1. *result* = AG
  2. $(A \rightarrow C$ and $A \rightarrow B)$ *result* = $ABCG$
  3. $(CG \rightarrow H$ and $CG \subseteq AGBC)$ *result* = $ABCGH$
  4. $(CG \rightarrow I$ and $CG \subseteq AGBCH)$ *result* = $ABCGHI$

- Is AG a super key?
- Is AG a key?
  1. Does $A^+ \rightarrow R$?
  2. Does $G^+ \rightarrow R$?

Example of Closure Computation

- Consider the "bad" relation `Students(Id, Name, AdvisorId, AdvisorName, FavouriteAdvisorId)`.
- What are the FDs that hold in this relation?

  `Id` $\rightarrow$ `Name`

  `Id` $\rightarrow$ `FavouriteAdvisorId`

  `AdvisorId` $\rightarrow$ `AdvisorName`

- To compute the key for this relation,
  1. Compute the closures for all sets of attributes.
  2. Find the minimal set of attributes whose closure is the set of all attributes.