CS/MATH 4414 Homework #4

Your task here is to typeset the following two pages of mathematics. You may assume all fonts are 10pt fonts, and the TEX parameter settings below. Otherwise you should match all fonts, indentations, alignments, spacing, etc. as closely as you can.

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\magnification=1095
%\hsize=5.41in \vsize=7.5in     %for magnification=1200
 \hsize=5.91in \vsize=8.18in    %for magnification=1095
\parindent=30pt \parskip=4pt plus 4pt minus 4pt
\baselineskip=13pt plus 2pt minus 1pt \lineskiplimit=2pt
\lineskip=2pt plus 2pt \tolerance=800 %\raggedright
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Approximation in Normed Linear Spaces

Definition. A vector space V with an inner product $\langle u, v\rangle$ is called an *inner product space*. A vector space V with a norm $\|x\|$ is called a *normed linear space*. A normed linear space is *strictly convex* if $\|x\| = \|y\| = \|\frac{1}{2}(x+y)\| = 1 \Rightarrow x = y$.

Theorem. Every finite dimensional subspace S of a normed linear space V contains a point closest to an arbitrary point $x \in V$. If V is strictly convex, then there is a unique closest point in S to x.

Applications of above theorem:

1. $V = C[a, b]$, $S = \mathcal{P}_n = \{\text{polynomials of degree} \le n\}$. For $f \in C[a, b]$, there exists a polynomial $P(x)$ of degree $\le n$ which minimizes

$$\|f - P\|_\infty = \max_{a \le x \le b} |f(x) - P(x)|$$

or

$$\|f - P\|_2 = \left(\int_a^b |f(x) - P(x)|^2 dx \right)^{1/2}$$

or

$$\|f - P\|_1 = \int_a^b |f(x) - P(x)|\, dx.$$

The "best" approximation P is unique only for the strictly convex norm $\|\cdot\|_2$.

2. $V = C[a, b]$, $S = \{\text{trigonometric polynomials of degree} \le n\}$. For $f \in C[a, b]$, there exists a trigonometric polynomial

$$T_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

of degree $\le n$ which minimizes $\|f - T_n\|_\infty$, or $\|f - T_n\|_2$, or $\|f - T_n\|_1$.

3. $V = C[a, b]$, $S = \{\text{continuous functions which are linear in each subinterval } [a+ih, a+(i+1)h],\ i = 0, \ldots, n-1,\ h = (b-a)/n\}$. For $f \in C[a, b]$, there exists $p \in S$ which minimizes $\|f - p\|_\infty$, or $\|f - p\|_2$, or $\|f - p\|_1$. Note that this p normally does not interpolate f at the nodes $a + ih$.

Theorem. Let V be an inner product space, and $S \subset V$ a finite dimensional subspace with orthonormal basis $\varphi_1, \ldots, \varphi_n$. Then for any point $f \in V$, the unique closest point in S to f is given by

$$P_S(f) = \sum_{i=1}^{n} \langle f, \varphi_i\rangle\, \varphi_i.$$

$P_S(f)$ is called the *projection* of f onto S, and $\langle f, \varphi_i\rangle$ are called *Fourier coefficients*.

Proof. Let $v = \sum_{i=1}^{n} \alpha_i \varphi_i$ be an arbitrary point in S. Then

$$\begin{aligned}\langle f - v, f - v\rangle &= \langle f - P_S(f) + P_S(f) - v, f - P_S(f) + P_S(f) - v\rangle \\ &= \langle f - P_S(f), f - P_S(f)\rangle + 2\,\langle f - P_S(f), P_S(f) - v\rangle + \langle P_S(f) - v, P_S(f) - v\rangle\end{aligned}$$

and

$$\langle f - P_S(f), P_S(f) - v \rangle = \left\langle f - \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i, \sum_{j=1}^{n} \beta_j \varphi_j \right\rangle = \sum_{j=1}^{n} \beta_j \left\langle f - \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i, \varphi_j \right\rangle$$
$$= \sum_{j=1}^{n} \beta_j \langle f - \langle f, \varphi_j \rangle \varphi_j, \varphi_j \rangle = \sum_{j=1}^{n} \beta_j \left(\langle f, \varphi_j \rangle - \langle f, \varphi_j \rangle \langle \varphi_j, \varphi_j \rangle \right)$$
$$= 0.$$

Therefore

$$\|f - P_S(f)\|^2 = \langle f - P_S(f), f - P_S(f) \rangle \leq \langle f - v, f - v \rangle = \|f - v\|^2$$

with equality $\Leftrightarrow \langle P_S(f) - v, P_S(f) - v \rangle = \|P_S(f) - v\|^2 = 0 \Leftrightarrow v = P_S(f)$. That is, $P_S(f)$ is the unique closest point in S to f. QED

Corollary. $f - P_S(f)$ is orthogonal to S.

Corollary (Parseval's Inequality). $\sum_{i=1}^{n} \langle f, \varphi_i \rangle^2 \leq \langle f, f \rangle$.

Note: if the basis $\varphi_1, \ldots, \varphi_n$ is merely orthogonal, then

$$P_S(f) = \sum_{i=1}^{n} \frac{\langle f, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle} \varphi_i$$

and the Fourier coefficients are $\langle f, \varphi_i \rangle / \langle \varphi_i, \varphi_i \rangle$. The projection operator $P_S(f)$ is an idempotent homomorphism $P_S : V \to V$ ($P_S \circ P_S = P_S$), and $\|P_S\| = 1$.