CUBIC SPLINES (an elementary approach)

The Weierstrass Approximation Theorem says that polynomials converge to $f \in C[a,b]$, but it does not say that interpolating polynomials converge. The Bernstein and Runge examples show that interpolating polynomials of higher and higher degree are not necessarily more accurate. Consider the error in polynomial interpolation:

$$f^{(n+1)}(\xi)\prod_{i=0}^{n}(x-x_i)/(n+1)!.$$

Since derivatives of f are usually an unknown quantity, the surest way to make the error small is to make the interval $[a,b]$ containing x_0, …, x_n small. Now the original interval $[a,b]$ is usually large, but one can interpolate on small subintervals, getting a *piecewise polynomial* approximation. The simplest case is a piecewise linear approximation, which is just a broken line.

Piecewise Hermite cubic. f is to be approximated by a piecewise cubic polynomial $g(x)$ with the properties that $g(x_i) = f(x_i)$, $g'(x_i) = f'(x_i)$, and $g(x)$ is a cubic polynomial $P_i(x)$ on each interval $[x_i, x_{i+1}]$, where $x_0 < x_1 < \cdots < x_n$. Newton's form for the Hermite cubic $P_i(x)$ in $[x_i, x_{i+1}]$ interpolates at x_i, x_i, x_{i+1}, x_{i+1} and is given by

$$P_i(x) = f[x_i] + f[x_i,x_i](x-x_i) + f[x_i,x_i,x_{i+1}](x-x_i)^2 + f[x_i,x_i,x_{i+1},x_{i+1}](x-x_i)^2(x-x_{i+1}).$$

Let $f_i = f(x_i)$, $s_i = f'(x_i)$, and rewrite $P_i(x)$ in terms of powers of $(x - x_i)$:

$$P_i(x) = c_{1,i} + c_{2,i}(x-x_i) + c_{3,i}(x-x_i)^2 + c_{4,i}(x-x_i)^3,$$

where

$$c_{1,i} = f_i, \qquad c_{2,i} = s_i, \qquad c_{3,i} = \frac{f[x_i,x_{i+1}] - s_i}{\Delta x_i} - c_{4,i}\Delta x_i, \qquad c_{4,i} = \frac{s_i + s_{i+1} - 2f[x_i,x_{i+1}]}{(\Delta x_i)^2}.$$

Note that $g(x)$ is C^1 on $[x_0, x_n]$, since $P_i'(x_{i+1}) = P_{i+1}'(x_{i+1}) = s_{i+1}$. $g(x)$ matches both f and f', so it is a good approximation to f, but it is not as "smooth" as it could be. By choosing the s_i, it is possible to construct a piecewise cubic which is C^2 and interpolates f. A C^2 curve is esthetically nicer than a C^1 curve; draftsmen can even "see" C^2 and C^3 discontinuities.

A C^2 piecewise cubic polynomial is called a *cubic spline*. In general,

Definition. A spline of degree m with nodes $x_0 < x_1 < \cdots < x_n$ is a C^{m-1} function which is a polynomial of degree $\le m$ in $(-\infty, x_0)$, (x_0, x_1), …, (x_{n-1}, x_n), (x_n, ∞). A natural spline of degree $2k+1$ is a spline of degree $2k+1$ which is a polynomial of degree $\le k$ in $(-\infty, x_0)$ and (x_n, ∞).

The basic result is

Theorem. Let $0 \le k \le n$, $x_0 < x_1 < \cdots < x_n$. Then for any set of values y_0, …, y_n $\exists$ a unique natural spline $S(x)$ of degree $2k+1$ with nodes x_0, …, x_n such that $S(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

Proof. Greville, *Theory and Applications of Spline Functions*, 1969.

Construction of a cubic spline (using first derivatives): In each interval $[x_i, x_{i+1}]$ the spline $g(x)$ has the form $P_i(x) = c_{1,i} + c_{2,i}(x - x_i) + c_{3,i}(x - x_i)^2 + c_{4,i}(x - x_i)^3$. The C^2 requirement means $P''_{i-1}(x_i) = P''_i(x_i) \Leftrightarrow 2c_{3,i-1} + 6c_{4,i-1}\Delta x_{i-1} = 2c_{3,i}$. Using the expressions for $c_{3,i}$, $c_{4,i}$ in terms of f_i, s_i (the s_i are now unspecified), this becomes

$$\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_{i-1}s_{i+1} = 3\big(\Delta x_{i-1}f[x_i, x_{i+1}] + \Delta x_i f[x_{i-1}, x_i]\big),\ i = 1, \ldots, n-1.$$

These are $n - 1$ linear equations in the $n + 1$ unknowns s_0, …, s_n. By specifying s_0, s_n, these become $n - 1$ equations in $n - 1$ unknowns, which have a unique solution since the coefficient matrix

$$\begin{pmatrix} 2(\Delta x_0 + \Delta x_1) & \Delta x_0 & 0 & 0 & \cdots \\ \Delta x_2 & 2(\Delta x_1 + \Delta x_2) & \Delta x_1 & 0 & \cdots \\ 0 & \Delta x_3 & 2(\Delta x_2 + \Delta x_3) & \Delta x_2 & \cdots \\ 0 & 0 & \Delta x_4 & 2(\Delta x_3 + \Delta x_4) & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is strictly row diagonally dominant.

In summary, given $x_0 < x_1 < \cdots < x_n$ and $f(x_i)$, $i = 0, 1, \ldots, n$:

(1) Choose s_0, s_n (ideally $s_0 = f'(x_0)$, $s_n = f'(x_n)$, the *complete spline interpolant*).

(2) Solve the tridiagonal system of linear equations for s_1, …, s_{n-1}.

(3) Construct the piecewise Hermite cubic $g(x)$ using $f(x_i)$ and s_i.

(4) Then $g(x)$ is a cubic spline on $[x_0, x_n]$ interpolating f at x_0, …, x_n.

Construction of a cubic spline (using second derivatives): Let $g(x)$ be a piecewise cubic given by $P_i(x)$ on $[x_i, x_{i+1}]$, with $g''(x_i\pm) = s_i$, $i = 0, 1, \ldots, n$. Note that here the unknowns s_i are second derivatives. Since $P_i(x)$ is a cubic, $P''_i(x) = s_i(x_{i+1} - x)/\Delta x_i + s_{i+1}(x - x_i)/\Delta x_i$ is linear. Integrating twice and requiring that $P_i(x_i) = f(x_i) = f_i$, $P_i(x_{i+1}) = f(x_{i+1}) = f_{i+1}$ yields

$$\begin{aligned} P_i(x) = \frac{s_i}{6\Delta x_i}(x_{i+1} - x)^3 + \frac{s_{i+1}}{6\Delta x_i}(x - x_i)^3 + \left(\frac{f_{i+1}}{\Delta x_i} - \frac{s_{i+1}\Delta x_i}{6}\right)(x - x_i) \\ + \left(\frac{f_i}{\Delta x_i} - \frac{s_i \Delta x_i}{6}\right)(x_{i+1} - x), \qquad i = 0, 1, \ldots, n-1. \end{aligned}$$

So far the pieces $P_i(x)$ and their second derivatives match at the nodes. The first derivatives must also match, so another condition is $P'_{i-1}(x_i) = P'_i(x_i)$, $i = 1, \ldots, n-1$. This results in the following system of $n - 1$ equations in the $n + 1$ unknowns s_0, …, s_n:

$$\Delta x_{i-1}s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_i s_{i+1} = 6\big(f[x_i, x_{i+1}] - f[x_{i-1}, x_i]\big), \qquad i = 1, \ldots, n-1.$$

Choosing s_0 and s_n uniquely determines the other s_i, since the coefficient matrix of the resulting linear system is strictly row diagonally dominant. The choice $s_0 = s_n = 0$ gives a *natural spline*.

In summary, given $x_0 < x_1 < \cdots < x_n$ and $f(x_i) = f_i$, $i = 0, 1, \ldots, n$:

(1) Choose $s_0 = s_n = 0$.

(2) Solve the tridiagonal linear system for s_i, $i = 1, \ldots, n-1$.

(3) Using these s_i, $g(x)$ given by $P_i(x)$ in $[x_i, x_{i+1}]$ is the unique natural cubic spline with nodes $x_0, x_1, \ldots, x_n$ interpolating f at $x_0, x_1, \ldots, x_n$.

Theorem. Let $f \in C^2[x_0, x_n]$, $x_0 < x_1 < \cdots < x_n$, $S(x)$ be the natural cubic spline interpolating f at $x_0, \ldots, x_n$, and let $g \in C^2[x_0, x_n]$ also interpolate f at $x_0, \ldots, x_n$. Then

$$\int_{x_0}^{x_n} \big(g''(x)\big)^2 dx \geq \int_{x_0}^{x_n} \big(S''(x)\big)^2 dx$$

with equality if and only if $g = S$.

Proof. $\int \big[g''(x) - S''(x)\big]^2 dx = \int \big(g''(x)\big)^2 dx - 2\int \big[g''(x) - S''(x)\big] S''(x) dx - \int \big(S''(x)\big)^2 dx$. The inequality will follow if the middle term is zero.

$$\begin{aligned}
\int_{x_0}^{x_n} (g'' - S'')S'' &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (g'' - S'')S'' \\
&= \sum_{i=0}^{n-1} S''(x)\big(g'(x) - S'(x)\big)\Big|_{x_i}^{x_{i+1}} - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \big(g'(x) - S'(x)\big) S'''(x) dx \\
&= S''(x_n)\big(g'(x_n) - S'(x_n)\big) - S''(x_0)\big(g'(x_0) - S'(x_0)\big) \\
&\qquad - \sum_{i=0}^{n-1} \alpha_i \int_{x_i}^{x_{i+1}} \big(g'(x) - S'(x)\big) dx
\end{aligned}$$

since $S'''(x)$ is a constant α_i on (x_i, x_{i+1}).

$$\int_{x_i}^{x_{i+1}} \big(g'(x) - S'(x)\big) dx = g(x) - S(x)\Big|_{x_i}^{x_{i+1}} = 0$$

since both g and S interpolate f at $x_0, \ldots, x_n$. Also $S''(x_0) = S''(x_n) = 0$ since S is a natural cubic spline. Hence $\int \big(g'' - S''\big)^2 = \int (g'')^2 - \int (S'')^2 \geq 0 \Rightarrow \int (g'')^2 \geq \int (S'')^2$.

There is equality $\Leftrightarrow \int \big(g'' - S''\big)^2 = 0 \Leftrightarrow g'' - S'' = 0$ since g'' and S'' are continuous. Now $g'' = S'' \Rightarrow g(x) = S(x) + c_1 x + c_2$. But $g(x_0) = S(x_0)$, $g(x_1) = S(x_1)$, $x_1 \neq x_0 \Rightarrow c_1 x_0 + c_2 = 0$, $c_1 x_1 + c_2 = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow g(x) = S(x)$. Q. E. D.

Corollary. Let $f \in C^2[x_0, x_n]$, $x_0 < x_1 < \cdots < x_n$, $S(x)$ be the complete cubic spline interpolant to f at $x_0, \ldots, x_n$, with $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$, and let $g \in C^2[x_0, x_n]$ also interpolate f at $x_0, \ldots, x_n$, with $g'(x_0) = f'(x_0)$, $g'(x_n) = f'(x_n)$. Then

$$\int_{x_0}^{x_n} \big(g''(x)\big)^2 dx \geq \int_{x_0}^{x_n} \big(S''(x)\big)^2 dx$$

with equality if and only if $g = S$.

Theorem. Let $f \in C^2[a,b]$, $S(x)$ be the natural cubic spline interpolating f at $a = x_0 < x_1 < \cdots < x_n = b$, and $h = \max\limits_{0 \le i \le n-1}(x_{i+1} - x_i)$. Then

$$\|f - S\|_\infty \le h^{3/2}\,\|f''\|_2 \qquad \text{and} \qquad \|f' - S'\|_\infty \le h^{1/2}\,\|f''\|_2\,.$$

Proof. Let $x \in [a,b]$. x is in some $[x_i, x_{i+1}]$, and since $f(t) - S(t)$ is zero at x_i and x_{i+1}, $f'(z) - S'(z) = 0$ for some $z \in (x_i, x_{i+1})$ by Rolle's Theorem. Then

$$\int_z^x \big[f''(t) - S''(t)\big]\,dt = f'(t) - S'(t)\Big|_z^x = f'(x) - S'(x).$$

Using the Cauchy-Schwarz Inequality,

$$\begin{aligned}|f'(x) - S'(x)| &= \left|\int_z^x \big[f''(t) - S''(t)\big]\cdot 1dt\right| \le \left|\int_z^x \big[f''(t) - S''(t)\big]^2 dt\right|^{1/2} \left|\int_z^x 1^2 dt\right|^{1/2} \\ &\le \left|\int_z^x \big[f''(t) - S''(t)\big]^2 dt\right|^{1/2} h^{1/2}.\end{aligned}$$

From the previous theorem, with $g = f$, $\int_a^b \big[f''(t) - S''(t)\big]^2 dt = \int_a^b f''(t)^2 dt - \int_a^b S''(t)^2 dt \le \int_a^b f''(t)^2 dt$. Since z and x are in $[a,b]$,

$$|f'(x) - S'(x)| \le \left(\int_a^b f''(t)^2 dt\right)^{1/2} h^{1/2} = \|f''\|_2\, h^{1/2}.$$

Finally, $f(x) - S(x) = \int_{x_i}^x \big[f'(t) - S'(t)\big]\,dt$, so

$$\begin{aligned}|f(x) - S(x)| &= \left|\int_{x_i}^x \big[f'(t) - S'(t)\big]\,dt\right| \\ &\le \int_{x_i}^x \max_{[a,b]} |f'(\tau) - S'(\tau)|dt = \max_{[a,b]} |f'(\tau) - S'(\tau)|(x - x_i) \\ &\le \|f''\|_2\, h^{1/2}(x - x_i) \le \|f''\|_2\, h^{3/2}.\end{aligned}$$

Q. E. D.

Theorem (deBoor, 1978). Let $f \in C^4[a,b]$, $S(x)$ be the complete cubic spline interpolating f at $a = x_0 < x_1 < \cdots < x_n = b$, $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$, and $h = \max\limits_{0 \le i \le n-1}(x_{i+1} - x_i)$. Then

$$\left\|f^{(k)} - S^{(k)}\right\|_\infty = \mathcal{O}\big(h^{4-k}\big), \qquad k = 0, 1, 2.$$

B-SPLINES (Carl deBoor)

Definition. Let $\xi = \left(\xi_i\right)_1^{\ell+1}$ be a strictly increasing sequence of points, $k > 0$, and $P_1, \ldots, P_\ell$ a sequence of ℓ polynomials each of order k (degree $< k$). The corresponding *piecewise polynomial* f of order k is defined by

$$f(x) = P_i(x), \qquad \xi_i < x < \xi_{i+1}; \quad i = 1, \ldots, \ell.$$

ξ_i are called the *breakpoints* of f. By convention,

$$f(x) = \begin{cases} P_1(x), & x \le \xi_1, \\ P_\ell(x), & x \ge \xi_{\ell+1}, \end{cases} \qquad \text{and} \qquad f(\xi_i) = f(\xi_i+) \quad \text{right continuous.}$$

$\mathcal{P}_{k,\xi} = \{\text{piecewise polynomial functions of order } k \text{ with breakpoint sequence } \left(\xi_i\right)_1^{\ell+1}\}$, and dim $\mathcal{P}_{k,\xi} = k\ell$.

Let $\nu = \left(\nu_i\right)_2^{\ell}$ be a vector of nonnegative integers, related to the jump conditions

$$\text{jump}_{\xi_i} D^{j-1} f = 0 \text{ for } j = 1, \ldots, \nu_i \text{ and } i = 2, \ldots, \ell.$$

$\mathcal{P}_{k,\xi,\nu} = \{f \in \mathcal{P}_{k,\xi} \mid |f \text{ satisfies the above jump conditions}\}$ is a subspace of $\mathcal{P}_{k,\xi}$ with dimension

$$\sum_{i=1}^{\ell} k - \nu_i \qquad (\nu_1 = 0).$$

A basis for $\mathcal{P}_{k,\xi}$ is

$$\phi_{ij} = \begin{cases} (x - \xi_1)^j / j!, & i = 1, \\ (x - \xi_i)_+^j / j!, & i = 2, \ldots, \ell, \end{cases} \qquad j = 0, \ldots, k-1,$$

where

$$(x - \xi_i)_+^j = \begin{cases} 0, & x < \xi_i, \\ (x - \xi_i)^j, & x \ge \xi_i. \end{cases}$$

A basis for $\mathcal{P}_{k,\xi,\nu}$ is ϕ_{ij}, $j = \nu_i, \ldots, k-1$ and $i = 1, \ldots, \ell$. That these are bases follows from the fact that they have the right number of elements, and are independent since $\exists$ linear functionals λ_{ij} such that

$$\lambda_{ij} \phi_{rs} = \delta_{ir} \delta_{js}. \qquad \left[\lambda_{ij} f = \text{jump}_{\xi_i} D^j f.\right]$$

Definition. Let $t = (t_i)$ be a nondecreasing sequence (finite, infinite, or biinfinite). The *ith B-spline of order k for the knot sequence t* is denoted by $B_{i,k,t}$ and is defined by

$$B_{i,k,t}(x) = (t_{i+k} - t_i)\,(\tau - x)_+^{k-1}\left[t_i, \ldots, t_{i+k}\right], \qquad \text{all } x \in E.$$

(The divided difference is applied to $(\tau - x)_+^{k-1}$ considered as a function of τ.) If k and t are understood, write B_i instead of $B_{i,k,t}$.

Properties of B-splines:

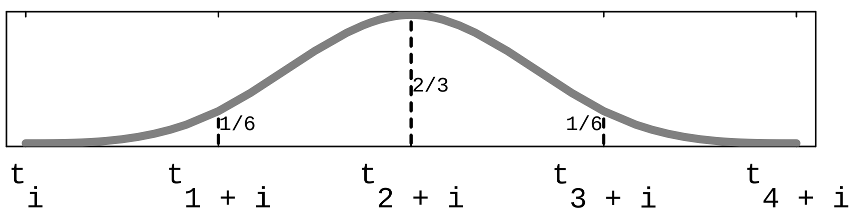
(i) $B_i(x) = 0$ for $x \notin \left[t_i, t_{i+k}\right]$.

(ii) $\sum_i B_i(x) = \sum_{i=r+1-k}^{s-1} B_i(x) = 1$ for all $t_r < x < t_s$.

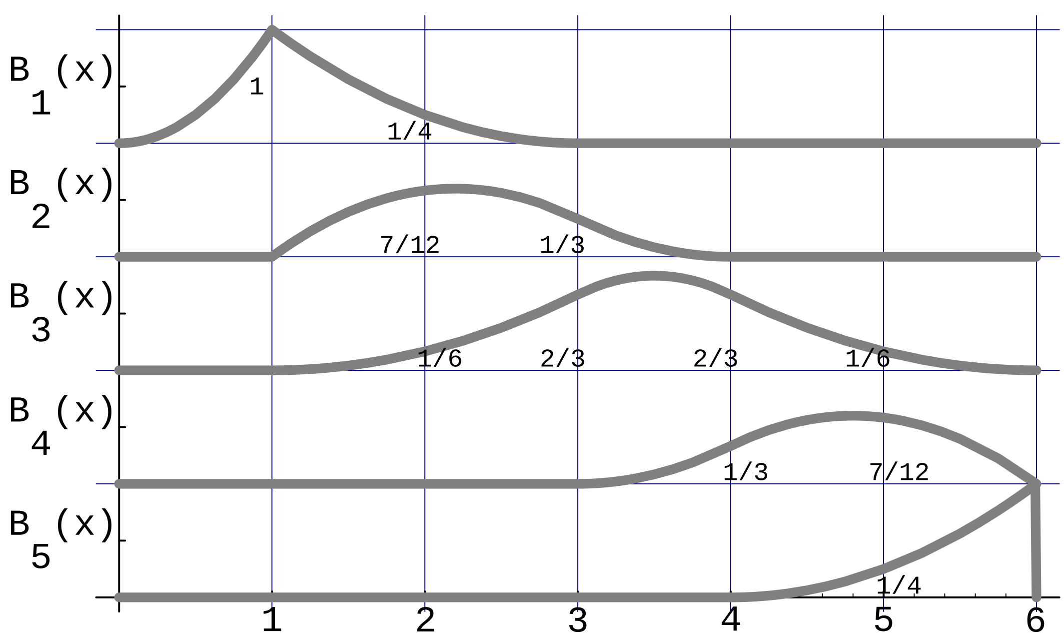
(iii) $B_i(x) > 0$ for $t_i < x < t_{i+k}$.

For t_i equally spaced a distance h apart, cubic ($k = 4$) B-splines are given by

$$B_i(x) = \frac{2}{3h^3}\left(\frac{1}{4}(x - t_i)_+^3 - (x - t_{i+1})_+^3 + \frac{3}{2}(x - t_{i+2})_+^3 - (x - t_{i+3})_+^3 + \frac{1}{4}(x - t_{i+4})_+^3\right).$$



Cubic B-spline $B_i(x)$ for equally spaces knots t_j.



Parabolic B-splines $B_{i,3}(x)$ for the knot sequence $t = (0, 1, 1, 3, 4, 6, 6, 6)$.

Definition. A spline function of order k with knot sequence t is any linear combination of B-splines of order k for the knot sequence t. The collection of all such functions is denoted by $\mathcal{S}_{k,t}$.

Theorem (Curry, Schönberg). For a given strictly increasing sequence $\xi = \big(\xi_i\big)_1^{\ell+1}$, and a given nonnegative integer sequence $\nu = (\nu_i)_2^\ell$ with $\nu_i \le k$, all i, set

$$n = k + \sum_{i=2}^{\ell} k - \nu_i = k\ell - \sum_{i=2}^{\ell} \nu_i = \dim \mathcal{P}_{k,\xi,\nu}$$

and let $t = \big(t_i\big)_1^{n+k}$ be any nondecreasing sequence so that

(i) $t_1 \le t_2 \le \cdots \le t_k \le \xi_1$ and $\xi_{\ell+1} \le t_{n+1} \le t_{n+2} \le \cdots \le t_{n+k}$;

(ii) for $i = 2, \ldots, \ell$, the number ξ_i occurs exactly $k - \nu_i$ times in t.

Then the sequence $B_1, \ldots, B_n$ of B-splines of order k for the knot sequence t is a basis for $\mathcal{P}_{k,\xi,\nu}$, considered as functions on $\big[t_k, t_{n+1}\big]$, i.e.,

$$\mathcal{S}_{k,t} = \mathcal{P}_{k,\xi,\nu} \qquad \text{on } \big[t_k, t_{n+1}\big].$$

NOTE: number of continuity conditions at ξ +
number of knots at ξ $= k$

Proof. From the definition of divided differences, for any sufficiently smooth function g $\exists$ constants d_i, …, d_{i+k} such that

$$g\big[t_i, \ldots, t_{i+k}\big] = \sum_{r=i}^{i+k} d_r g^{(j_r)}(t_r),$$

with $j_r = \max\{s \mid r - s \ge i,\ t_{r-s} = t_r\}$, $r = i, \ldots, i+k$. Thus

$$B_i(x) = (t_{i+k} - t_i) \sum_{r=i}^{i+k} d_r \big(t_r - x\big)_+^{k-1-j_r} (k-1)!/(k-1-j_r)!,$$

which is clearly a piecewise polynomial function of order k with breakpoints at t_i, …, t_{i+k}, i.e., at some of the points ξ_2, …, ξ_ℓ (and possibly at some other points outside $(\xi_1, \xi_{\ell+1})$, but these don't matter). B_i has a jump in its s-th derivative at the breakpoint ξ_j only if for some $r \in [i, i+k]$, we have $\xi_j = t_r$ and $k-1-j_r = s$. Since j_r counts the number of t_m's equal to t_r and with $i \le m < r$, it follows that j_r must be less than $k - \nu_j$ which is the total number of t_m's equal to $\xi_j = t_r$ by construction of t. This says that always $s \ge \nu_j$, and so

$$\text{jump}_{\xi_j} D^m B_i = 0 \qquad \text{for } m = 0, \ldots, \nu_j - 1.$$

Therefore $B_i \in \mathcal{P}_{k,\xi,\nu}$, all i.

Since there are n B_i's and $\dim \mathcal{P}_{k,\xi,\nu} = n$, it suffices to show that the sequence $\big(B_i\big)_1^n$ is linearly independent. This follows from:

Lemma (deBoor, Fix, 1973). Let λ_i be the linear functional given by the rule

$$\lambda_i f = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi^{(k-1-r)}(\tau_i)\, D^r f(\tau_i),$$

all f, with $\psi(t) = \big(t_{i+1} - t\big) \cdots \big(t_{i+k-1} - t\big)\big/(k-1)!$, and τ_i some arbitrary point in the open interval (t_i, t_{i+k}). Then

$$\lambda_i\, B_j = \delta_{ij}, \qquad \text{all } j.$$

Q. E. D.

B-SPLINE INTERPOLATION.

Let $t = \big(t_i\big)_1^{n+k}$ be a nondecreasing knot sequence with $t_i < t_{i+k}$, all i, and $\big(B_i\big)_1^n$ the corresponding B-splines of order k. The span $\mathcal{S}_{k,t}$ of $B_1, \ldots, B_n$ is n-dimensional. Given a strictly increasing sequence $\tau = \big(\tau_i\big)_1^n$ and function g, the problem is to find $f \in \mathcal{S}_{k,t}$ such that $f(\tau_i) = g(\tau_i)$ $\forall i$. Or, find spline coefficients α_j such that

$$\sum_{j=1}^{n} \alpha_j B_j(\tau_i) = g(\tau_i), \qquad i = 1, \ldots, n.$$

Theorem (Schönberg, Whitney). The matrix $\big(B_j(\tau_i)\big)$ is invertible $\Leftrightarrow$

$$B_i(\tau_i) \neq 0, \qquad i = 1, \ldots, n,$$

i.e., $t_i < \tau_i < t_{i+k}$, all i.

Theorem (Karlin). The matrix $\big(B_j(\tau_i)\big)$ is totally positive (all minors ≥ 0).

Observation. $\big(B_j(\tau_i)\big)$ has bandwidth less than k if $t_i < \tau_i < t_{i+k}$, all i.

Spline (a.k.a. piecewise polynomial) representations.

Consider the case of a piecewise cubic polynomial $g(x)$ with breakpoints $\xi_1 < \xi_2 < \cdots < \xi_{n+1}$.

(1) Polynomial coefficients c_{ji}, $1 \le i \le n$, $1 \le j \le 4$. On each interval $[\xi_i, \xi_{i+1})$,

$$g(x) = P_i(x) = \sum_{j=1}^{4} c_{ji}(x-\xi_i)^{j-1},$$

and by convention

$$g(x) = \begin{cases} P_1(x), & x \le \xi_1 \\ P_n(x), & x \ge \xi_{n+1} \end{cases}.$$

(2) Truncated power basis

$$\phi_{ij} = \begin{cases} (x-\xi_1)^j/j!, & i=1, \\ (x-\xi_i)_+^j/j!, & i=2,\ldots,n, \end{cases} \qquad j=0,\ldots,3,$$

where

$$(x-\xi_i)_+^j = \begin{cases} 0, & x < \xi_i, \\ (x-\xi_i)^j, & x \ge \xi_i. \end{cases} \qquad g(x) = \sum_{\substack{1\le i\le n \\ 0 \le j \le 3}} \alpha_{ij}\phi_{ij}(x), \quad \xi_1 \le x \le \xi_{n+1}.$$

Proof. For $1 \le i \le n$ and $0 \le j \le 3$, $\phi_{ij}(x) \in \mathcal{P}_{4,\xi} = \{$piecewise cubic (order 4) polynomials with breakpoints $\xi_1, \ldots, \xi_{n+1}\}$. From (1), dim $\mathcal{P}_{4,\xi} = 4n$. The $4n$ vectors $\phi_{ij}(x)$, $1 \le i \le n$, $0 \le j \le 3$, are linearly independent. Therefore the ϕ_{ij} are a basis for $\mathcal{P}_{4,\xi}$, and any $g \in \mathcal{P}_{4,\xi}$ is a unique linear combination of the ϕ_{ij}. Q. E. D.

$$\mathcal{P}_{k,\xi,\nu} = \big\{\text{piecewise polynomials } g(x) \text{ of degree} \le k-1 \text{ with breakpoints } \xi_1 < \xi_2 < \cdots < \xi_{n+1} \\ \text{such that } g^{(\nu_i-1)}(x) \text{ exists at } \xi_i, \text{ and } 0 \le \nu_i \le k, \text{ for } 2 \le i \le n\big\}.$$

The special case $k=4$, $\nu=(2,\ldots,2)$ is called *piecewise Hermite cubics*. Let $h_i = \xi_{i+1}-\xi_i$,

$$c_0^i(x) = \frac{2}{h_i^3}(x-\xi_{i+1})^2\Big(x-\xi_i+\frac{h_i}{2}\Big), \quad \hat{c}_0^i(x) = \frac{1}{h_i^2}(x-\xi_i)(x-\xi_{i+1})^2,$$
$$c_1^i(x) = -\frac{2}{h_i^3}(x-\xi_i)^2\Big(x-\xi_{i+1}-\frac{h_i}{2}\Big), \quad \hat{c}_1^i(x) = \frac{1}{h_i^2}(x-\xi_i)^2(x-\xi_{i+1}).$$

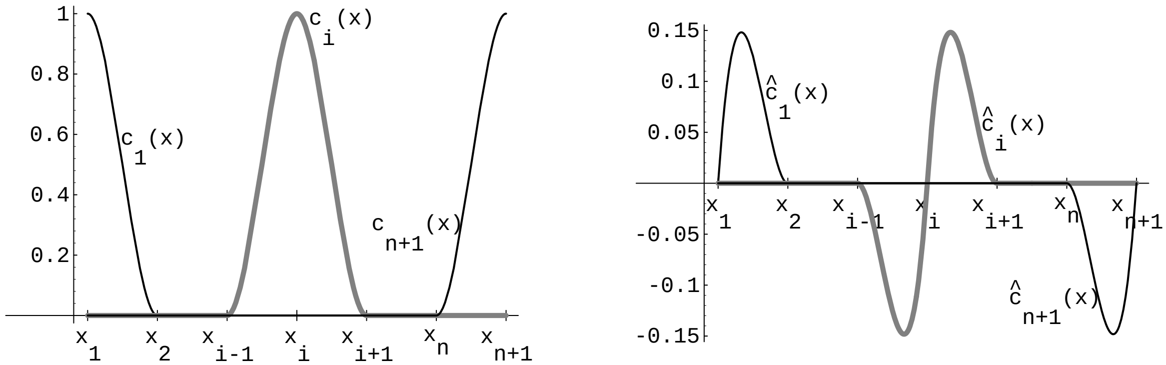
The C' piecewise Hermite cubics $c_i(x), \hat{c}_i(x)$ are defined by

$$c_1(x) \begin{cases} c_0^1(x) & \text{, on } [\xi_1,\xi_2] \\ 0 & \text{, on } [\xi_2,\xi_{n+1}] \end{cases}, \quad \hat{c}_1(x) = \begin{cases} \hat{c}_0^1(x) & \text{, on } [\xi_1,\xi_2] \\ 0 & \text{, on } [\xi_2,\xi_{n+1}] \end{cases},$$

for $i = 2, \ldots, n$,

$$c_i(x) = \begin{cases} 0 & \text{, on } [\xi_1,\xi_{i-1}] \\ c_1^{i-1}(x) & \text{, on } [\xi_{i-1},\xi_i] \\ c_0^i(x) & \text{, on } [\xi_i,\xi_{i+1}] \\ 0 & \text{, on } [\xi_{i+1},\xi_{n+1}] \end{cases}, \quad \hat{c}_i(x) = \begin{cases} 0 & \text{, on } [\xi_1,\xi_{i-1}] \\ \hat{c}_1^{i-1}(x) & \text{, on } [\xi_{i-1},\xi_i] \\ \hat{c}_0^i(x) & \text{, on } [\xi_i,\xi_{i+1}] \\ 0 & \text{, on } [\xi_{i+1},\xi_{n+1}] \end{cases},$$

$$c_{n+1}(x) \begin{cases} 0 & \text{, on } [\xi_1,\xi_n] \\ c_1^n(x) & \text{, on } [\xi_n,\xi_{n+1}] \end{cases}, \quad \hat{c}_{n+1}(x) = \begin{cases} 0 & \text{, on } [\xi_1,\xi_n] \\ \hat{c}_1^n(x) & \text{, on } [\xi_n,\xi_{n+1}] \end{cases}.$$



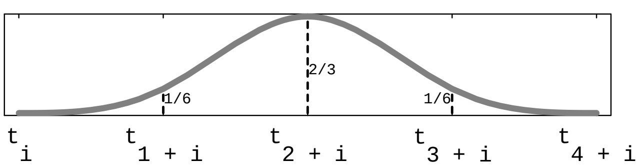
$c_i(\xi_j) = \delta_{ij}, \quad c_i'(\xi_j) = 0, \quad 1 \le i, j \le n+1. \qquad \hat{c}_i'(\xi_j) = \delta_{ij}, \quad \hat{c}_i(\xi_j) = 0, \quad 1 \le i, j \le n+1.$

(3) Piecewise Hermite cubic basis $c_i(x)$, $\hat{c}_i(x)$.

$$g(x) = \sum_{i=1}^{n+1} y_i c_i(x) + d_i \hat{c}_i(x).$$

Note that $g(\xi_i) = y_i$ and $g'(\xi_i) = d_i$.

Standard cubic splines correspond to the special case $k = 4$, $\nu = (3, 3, \ldots, 3)$ of $\mathcal{P}_{k,\xi,\nu}$. Basis functions $B_i(x)$, called B-splines, for this space look like



The points t_i are called knots, which are not identical with the breakpoints ξ_i. Precisely, given the breakpoint sequence ξ, a standard choice for the knot sequence t is $t_1 = t_2 = t_3 = t_4 = \xi_1$, $t_{i+3} = \xi_i$ for $i = 2, \ldots, n$, $t_{n+4} = t_{n+5} = t_{n+6} = t_{n+7} = \xi_{n+1}$. Then for any C^2 cubic spline $g \in \mathcal{P}_{4,\xi,3}$, $g(x) = \sum_{i=1}^{n+3} \alpha_i B_i(x)$, giving the

(4) B-spline basis $B_i(x)$.

The real power of B-splines lies in

Theorem. For any $k \ge 1$, breakpoint sequence $\xi_1 < \xi_2 < \cdots < \xi_{l+1}$, and smoothness sequence $\nu = (\nu_i)_2^l$ with all $\nu_i \le k$, there exists a knot sequence $t = (t_i)_1^{n+k}$, where

$$n = kl - \sum_{i=2}^{l} \nu_i,$$

such that the B-splines $B_1, \ldots, B_n$ of order k for the knot sequence t are a basis for $\mathcal{P}_{k,\xi,\nu}$ on $[t_k, t_{n+1}]$.

Cubic spline interpolants.

The B-spline representation $g(x) = \sum_{i=1}^{n+3} \alpha_i B_i(x)$ shows clearly there are $n+3$ degrees of freedom. Thus fixing $g(\xi_i) = f(\xi_i)$ for $i = 1, \ldots, n+1$, leaves 2 degrees of freedom. Some standard choices are:

(1) $g'(\xi_1) = f'(\xi_1),\ g'(\xi_{n+1}) = f'(\xi_{n+1})$: **complete** cubic spline;

(2) $g''(\xi_1) = g''(\xi_{n+1}) = 0$: **natural** cubic spline;

(3) $g'''(x)$ is continuous at ξ_2 and ξ_n : **not-a-knot** condition.

"The only good thing about a natural cubic spline is its name." (Carl deBoor).
