CUBIC SPLINES (an elementary approach)

The Weierstrass Approximation Theorem says that polynomials converge to \( f \in C[a,b] \), but it does not say that interpolating polynomials converge. The Bernstein and Runge examples show that interpolating polynomials of higher and higher degree are not necessarily more accurate. Consider the error in polynomial interpolation:

\[
    f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)/(n + 1)!.
\]

Since derivatives of \( f \) are usually an unknown quantity, the surest way to make the error small is to make the interval \([a,b]\) containing \( x_0, \ldots, x_n \) small. Now the original interval \([a,b]\) is usually large, but one can interpolate on small subintervals, getting a piecewise polynomial approximation. The simplest case is a piecewise linear approximation, which is just a broken line.

**Piecewise Hermite cubic.** \( f \) is to be approximated by a piecewise cubic polynomial \( g(x) \) with the properties that \( g(x_i) = f(x_i), g'(x_i) = f'(x_i) \), and \( g(x) \) is a cubic polynomial \( P_i(x) \) on each interval \([x_i, x_{i+1}]\), where \( x_0 < x_1 < \cdots < x_n \). Newton’s form for the Hermite cubic \( P_i(x) \) in \([x_i, x_{i+1}]\) interpolates at \( x_i, x_{i+1} \), and is given by

\[
    P_i(x) = f[x_i] + f[x_i, x_i](x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2(x - x_{i+1}).
\]

Let \( f_i = f(x_i), s_i = f'(x_i) \), and rewrite \( P_i(x) \) in terms of powers of \((x - x_i)\):

\[
    P_i(x) = c_{1,i} + c_{2,i}(x - x_i) + c_{3,i}(x - x_i)^2 + c_{4,i}(x - x_i)^3,
\]

where

\[
    c_{1,i} = f_i, \quad c_{2,i} = s_i, \quad c_{3,i} = \frac{f[x_i, x_{i+1}] - s_i}{\Delta x_i} - c_{4,i}\Delta x_i, \quad c_{4,i} = \frac{s_i + s_{i+1} - 2f[x_i, x_{i+1}]}{(\Delta x_i)^2}.
\]

Note that \( g(x) \) is \( C^1 \) on \([x_0, x_n]\), since \( P_i'(x_{i+1}) = P_{i+1}'(x_{i+1}) = s_{i+1} \). \( g(x) \) matches both \( f \) and \( f' \), so it is a good approximation to \( f \), but it is not as “smooth” as it could be. By choosing the \( s_i \), it is possible to construct a piecewise cubic which is \( C^2 \) and interpolates \( f \). A \( C^2 \) curve is esthetically nicer than a \( C^1 \) curve; draftsmen can even “see” \( C^2 \) and \( C^3 \) discontinuities.

A \( C^2 \) piecewise cubic polynomial is called a cubic spline. In general,

**Definition.** A spline of degree \( m \) with nodes \( x_0 < x_1 < \cdots < x_n \) is a \( C^{m-1} \) function which is a polynomial of degree \( \leq m \) in \((-\infty, x_0), (x_0, x_1), \ldots, (x_{n-1}, x_n), (x_n, \infty)\). A natural spline of degree \( 2k + 1 \) is a spline of degree \( 2k + 1 \) which is a polynomial of degree \( \leq k \) in \((-\infty, x_0)\) and \((x_n, \infty)\).

The basic result is

**Theorem.** Let \( 0 \leq k \leq n \), \( x_0 < x_1 < \cdots < x_n \). Then for any set of values \( y_0, \ldots, y_n \), \( \exists \) a unique natural spline \( S(x) \) of degree \( 2k + 1 \) with nodes \( x_0, \ldots, x_n \) such that \( S(x_i) = y_i \) for \( i = 0, 1, \ldots, n \).

Construction of a cubic spline (using first derivatives): In each interval \([x_i, x_{i+1}]\) the spline \(g(x)\) has the form \(P_i(x) = c_{1,i} + c_{2,i}(x-x_i) + c_{3,i}(x-x_i)^2 + c_{4,i}(x-x_i)^3\). The \(C^2\) requirement means \(P''_i(x_i) = P''_{i-1}(x_{i-1}) \iff 2c_{3,i-1} + 6c_{4,i-1}\Delta x_{i-1} = 2c_{3,i}\). Using the expressions for \(c_{3,i}, c_{4,i}\) in terms of \(f_i, s_i\) (the \(s_i\) are now unspecified), this becomes

\[
\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_{i-1} s_{i+1} = 3(\Delta x_{i-1} f[x_i, x_{i+1}] + \Delta x_i f[x_{i-1}, x_i]), \quad i = 1, \ldots, n-1.
\]

These are \(n-1\) linear equations in the \(n+1\) unknowns \(s_0, \ldots, s_n\). By specifying \(s_0, s_n\), these become \(n-1\) equations in \(n-1\) unknowns, which have a unique solution since the coefficient matrix

\[
\begin{pmatrix}
2(\Delta x_0 + \Delta x_1) & \Delta x_0 & 0 & 0 & \cdots \\
\Delta x_2 & 2(\Delta x_1 + \Delta x_2) & \Delta x_1 & 0 & \cdots \\
0 & \Delta x_3 & 2(\Delta x_2 + \Delta x_3) & \Delta x_2 & \cdots \\
0 & 0 & \Delta x_4 & 2(\Delta x_3 + \Delta x_4) & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

is strictly row diagonally dominant.

In summary, given \(x_0 < x_1 < \cdots < x_n\) and \(f(x_i), i = 0, 1, \ldots, n:\)

1. Choose \(s_0, s_n\) (ideally \(s_0 = f'(x_0), s_n = f'(x_n)\), the complete spline interpolant).
2. Solve the tridiagonal system of linear equations for \(s_1, \ldots, s_{n-1}\).
3. Construct the piecewise Hermite cubic \(g(x)\) using \(f(x_i)\) and \(s_i\).
4. Then \(g(x)\) is a cubic spline on \([x_0, x_n]\) interpolating \(f\) at \(x_0, \ldots, x_n\).

Construction of a cubic spline (using second derivatives): Let \(g(x)\) be a piecewise cubic given by \(P_i(x)\) on \([x_i, x_{i+1}]\), with \(g''(x_i \pm) = s_i, i = 0, 1, \ldots, n\). Note that here the unknowns \(s_i\) are second derivatives. Since \(P_i(x)\) is a cubic, \(P''_i(x) = s_i(x_{i+1} - x) / \Delta x_i + s_{i+1}(x - x_i) / \Delta x_i\) is linear. Integrating twice and requiring that \(P_i(x_i) = f(x_i) = f_i, P_i(x_{i+1}) = f(x_{i+1}) = f_{i+1}\) yields

\[
P_i(x) = \frac{s_i}{6\Delta x_i}(x_{i+1} - x)^3 + \frac{s_{i+1}}{6\Delta x_i}(x-x_i)^3 + \left(\frac{f_{i+1}}{\Delta x_i} - \frac{s_{i+1}\Delta x_i}{6}\right)(x-x_i) + \left(\frac{f_i}{\Delta x_i} - \frac{s_i\Delta x_i}{6}\right)(x_{i+1} - x), \quad i = 0, 1, \ldots, n-1.
\]

So far the pieces \(P_i(x)\) and their second derivatives match at the nodes. The first derivatives must also match, so another condition is \(P'_i(x_i) = P'_{i-1}(x_{i-1}), i = 1, \ldots, n-1\). This results in the following system of \(n-1\) equations in the \(n+1\) unknowns \(s_0, \ldots, s_n:\)

\[
\Delta x_{i-1}s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_i s_{i+1} = 6(f[x_i, x_{i+1}] - f[x_{i-1}, x_i]), \quad i = 1, \ldots, n-1.
\]

Choosing \(s_0\) and \(s_n\) uniquely determines the other \(s_i\), since the coefficient matrix of the resulting linear system is strictly row diagonally dominant. The choice \(s_0 = s_n = 0\) gives a natural spline.

In summary, given \(x_0 < x_1 < \cdots < x_n\) and \(f(x_i) = f_i, i = 0, 1, \ldots, n:\)

1. Choose \(s_0 = s_n = 0\).

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Theorem. Let \( f \in C^2[x_0, x_n], x_0 < x_1 < \cdots < x_n, S(x) \) be the natural cubic spline interpolating \( f \) at \( x_0, \ldots, x_n \), and let \( g \in C^2[x_0, x_n] \) also interpolate \( f \) at \( x_0, \ldots, x_n \). Then

\[
\int_{x_0}^{x_n} (g''(x))^2 \, dx \geq \int_{x_0}^{x_n} (S''(x))^2 \, dx
\]

with equality if and only if \( g = S \).

Proof. \( \int [g''(x) - S''(x)]^2 \, dx = \int (g''(x))^2 \, dx - 2 \int [g''(x) - S''(x)]S''(x) \, dx - \int (S''(x))^2 \, dx \). The inequality will follow if the middle term is zero.

\[
\int_{x_0}^{x_n} (g'' - S'')S'' = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (g'' - S'')S''
\]

\[
= \sum_{i=0}^{n-1} S''(x)\left(g'(x) - S'(x)\right)\bigg|_{x_i}^{x_{i+1}} - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (g'(x) - S'(x))S'''(x) \, dx
\]

\[
= S'''(x_n)(g'(x_n) - S'(x_n)) - S'''(x_0)(g'(x_0) - S'(x_0))
\]

\[
- \sum_{i=0}^{n-1} \alpha_i \int_{x_i}^{x_{i+1}} (g'(x) - S'(x)) \, dx
\]

since \( S'''(x) \) is a constant \( \alpha_i \) on \( (x_i, x_{i+1}) \).

\[
\int_{x_i}^{x_{i+1}} (g'(x) - S'(x)) \, dx = g(x) - S(x)\bigg|_{x_i}^{x_{i+1}} = 0
\]

since both \( g \) and \( S \) interpolate \( f \) at \( x_0, \ldots, x_n \). Also \( S''(x_0) = S''(x_n) = 0 \) since \( S \) is a natural cubic spline. Hence \( \int (g'' - S'')^2 = \int (g'')^2 - \int (S'')^2 \geq 0 \Rightarrow \int (g'')^2 \geq \int (S'')^2 \).

There is equality \( \iff \int (g'' - S'')^2 = 0 \iff g'' = S'' = 0 \) since \( g'' \) and \( S'' \) are continuous. Now \( g'' = S'' \Rightarrow g(x) = S(x) + c_1x + c_2 \). But \( g(x_0) = S(x_0), g(x_1) = S(x_1), x_1 \neq x_0 \Rightarrow c_1 x_0 + c_2 = 0, c_1 = c_2 = 0 \Rightarrow g(x) = S(x) \). Q. E. D.

Corollary. Let \( f \in C^2[x_0, x_n], x_0 < x_1 < \cdots < x_n, S(x) \) be the complete cubic spline interpolant to \( f \) at \( x_0, \ldots, x_n \), with \( S'(x_0) = f'(x_0), S'(x_n) = f'(x_n) \), and let \( g \in C^2[x_0, x_n] \) also interpolate \( f \) at \( x_0, \ldots, x_n \), with \( g'(x_0) = f'(x_0), g'(x_n) = f'(x_n) \). Then

\[
\int_{x_0}^{x_n} (g''(x))^2 \, dx \geq \int_{x_0}^{x_n} (S''(x))^2 \, dx
\]

with equality if and only if \( g = S \).
Theorem. Let \( f \in C^2[a,b] \), \( S(x) \) be the natural cubic spline interpolating \( f \) at \( a = x_0 < x_1 < \ldots < x_n = b \), and \( h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i) \). Then

\[
\|f - S\|_\infty \leq h^{3/2} \|f''\|_2 \quad \text{and} \quad \|f' - S'\|_\infty \leq h^{1/2} \|f''\|_2.
\]

Proof. Let \( x \in [a,b] \). \( x \) is in some \( [x_i, x_{i+1}] \), and since \( f(t) - S(t) \) is zero at \( x_i \) and \( x_{i+1} \), \( f'(z) - S'(z) = 0 \) for some \( z \in (x_i, x_{i+1}) \) by Rolle's Theorem. Then

\[
\int_{x_i}^{x} \left[ f''(t) - S''(t) \right] dt = f'(t) - S'(t) \bigg|_{x_i}^{x} = f'(x) - S'(x).
\]

Using the Cauchy-Schwarz Inequality,

\[
|f'(x) - S'(x)| = \left| \int_{x_i}^{x} [f''(t) - S''(t)] \cdot 1 \, dt \right| \leq \left( \int_{x_i}^{x} [f''(t) - S''(t)]^2 \, dt \right)^{1/2} \left( \int_{x_i}^{x} 1^2 \, dt \right)^{1/2}
\]

\[
\leq \left( \int_{x_i}^{x} [f''(t) - S''(t)]^2 \, dt \right)^{1/2} h^{1/2}.
\]

From the previous theorem, with \( g = f \), \( \int_{a}^{b} [f''(t) - S''(t)]^2 \, dt = f''_a f''(t)^2 \, dt - \int_{a}^{b} S''(t)^2 \, dt \leq f''_a f''(t)^2 \, dt \). Since \( z \) and \( x \) are in \([a,b]\),

\[
|f'(x) - S'(x)| \leq \left( \int_{a}^{b} f''(t)^2 \, dt \right)^{1/2} h^{1/2} = \|f''\|_2 h^{1/2}.
\]

Finally, \( f(x) - S(x) = \int_{x_i}^{x} [f'(t) - S'(t)] \, dt \), so

\[
|f(x) - S(x)| = \left| \int_{x_i}^{x} [f'(t) - S'(t)] \, dt \right|
\]

\[
\leq \int_{x_i}^{x} \max_{[a,b]} |f'(\tau) - S'(\tau)| \, d\tau = \max_{[a,b]} |f'(\tau) - S'(\tau)|(x - x_i)
\]

\[
\leq \|f''\|_2 h^{1/2} (x - x_i) \leq \|f''\|_2 h^{3/2}.
\]

Q. E. D.

Theorem (deBoor, 1978). Let \( f \in C^4[a,b] \), \( S(x) \) be the complete cubic spline interpolating \( f \) at \( a = x_0 < x_1 < \ldots < x_n = b \), \( S'(x_0) = f'(x_0) \), \( S'(x_n) = f'(x_n) \), and \( h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i) \). Then

\[
\|f^{(k)} - S^{(k)}\|_\infty = O(h^{4-k}), \quad k = 0, 1, 2.
\]

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**B-SPLINES (Carl deBoor)**

**Definition.** Let $\xi = (\xi_i)_{i=1}^{\ell+1}$ be a strictly increasing sequence of points, $k > 0$, and $P_1, \ldots, P_\ell$ a sequence of $\ell$ polynomials each of order $k$ (degree $< k$). The corresponding piecewise polynomial $f$ of order $k$ is defined by

$$f(x) = P_i(x), \quad \xi_i < x < \xi_{i+1}; \quad i = 1, \ldots, \ell.$$ 

$\xi_i$ are called the breakpoints of $f$. By convention,

$$f(x) = \begin{cases} 
  P_1(x), & x \leq \xi_1, \\
  P_\ell(x), & x \geq \xi_{\ell+1}, 
\end{cases} \quad \text{and} \quad f(\xi_i) = f(\xi_i+) \quad \text{right continuous.}$$

$\mathcal{P}_{k,\xi} = \{\text{piecewise polynomial functions of order } k \text{ with breakpoint sequence } (\xi_i)_{i=1}^{\ell+1}\}$, and dim $\mathcal{P}_{k,\xi} = k\ell$.

Let $\nu = (\nu_i)_2^\ell$ be a vector of nonnegative integers, related to the jump conditions

$$\text{jump}_{\xi_i} D^j f = 0 \text{ for } j = 1, \ldots, \nu_i \text{ and } i = 2, \ldots, \ell.$$ 

$\mathcal{P}_{k,\xi,\nu} = \{f \in \mathcal{P}_{k,\xi} \mid |f \text{ satisfies the above jump conditions}\}$ is a subspace of $\mathcal{P}_{k,\xi}$ with dimension

$$\sum_{i=1}^{\ell} k - \nu_i \quad (\nu_1 = 0).$$

A basis for $\mathcal{P}_{k,\xi}$ is

$$\phi_{ij} = \begin{cases} 
  (x - \xi_i)^j/j!, & i = 1, \\
  (x - \xi_i)^j_+ / j!, & i = 2, \ldots, \ell, 
\end{cases} \quad j = 0, \ldots, k - 1,$$

where

$$(x - \xi_i)^j_+ = \begin{cases} 
  0, & x < \xi_i, \\
  (x - \xi_i)^j, & x \geq \xi_i.
\end{cases}$$

A basis for $\mathcal{P}_{k,\xi,\nu}$ is $\phi_{ij}$, $j = \nu_i, \ldots, k - 1$ and $i = 1, \ldots, \ell$. That these are bases follows from the fact that they have the right number of elements, and are independent since $\exists$ linear functionals $\lambda_{ij}$ such that

$$\lambda_{ij} \phi_{rs} = \delta_{ir} \delta_{js}, \quad [\lambda_{ij} f = \text{jump}_{\xi_i} D^j f].$$

**Definition.** Let $t = (t_i)$ be a nondecreasing sequence (finite, infinite, or biinfinite). The $i$th B-spline of order $k$ for the knot sequence $t$ is denoted by $B_{i,k,t}$ and is defined by

$$B_{i,k,t}(x) = (t_{i+k} - t_i) (\tau - x)^{k-1} \left[ t_i, \ldots, t_{i+k} \right], \quad \text{all } x \in E.$$ 

(The divided difference is applied to $(\tau - x)^{k-1}$ considered as a function of $\tau$.) If $k$ and $t$ are understood, write $B_i$ instead of $B_{i,k,t}$.
Properties of B-splines:

(i) \( B_i(x) = 0 \) for \( x \notin [t_i, t_{i+k}] \).

(ii) \( \sum_i B_i(x) = \sum_{i=r+1-k}^{s-1} B_i(x) = 1 \) for all \( t_r < x < t_s \).

(iii) \( B_i(x) > 0 \) for \( t_i < x < t_{i+k} \).

For \( t_i \) equally spaced a distance \( h \) apart, cubic \((k = 4)\) B-splines are given by

\[
B_i(x) = \frac{2}{3h^3} \left( \frac{1}{4}(x - t_i)^3_+ - (x - t_{i+1})^3_+ + \frac{3}{2}(x - t_{i+2})^3_+ - (x - t_{i+3})^3_+ + \frac{1}{4}(x - t_{i+4})^3_+ \right).
\]

Cubic B-spline \( B_i(x) \) for equally spaces knots \( t_j \).

Parabolic B-splines \( B_{i,3}(x) \) for the knot sequence \( t = (0, 1, 1, 3, 4, 6, 6, 6) \).
**Definition.** A spline function of order \( k \) with knot sequence \( t \) is any linear combination of B-splines of order \( k \) for the knot sequence \( t \). The collection of all such functions is denoted by \( S_{k,t} \).

**Theorem (Curry, Schönberg).** For a given strictly increasing sequence \( \xi = (\xi_i)_{i=1}^{\ell+1} \) and a given nonnegative integer sequence \( \nu = (\nu_i)_{i=2}^{\ell} \) with \( \nu_i \leq k \), all \( i \), set

\[
\begin{align*}
n &= k + \sum_{i=2}^{\ell} k - \nu_i = k\ell - \sum_{i=2}^{\ell} \nu_i = \dim P_{k,\xi,\nu} \\
\end{align*}
\]

and let \( t = (t_i)_{i=1}^{n+k} \) be any nondecreasing sequence so that

(i) \( t_1 \leq t_2 \leq \cdots \leq t_k \leq \xi_1 \) and \( \xi_{\ell+1} \leq t_{n+1} \leq t_{n+2} \leq \cdots \leq t_{n+k} \);

(ii) for \( i = 2, \ldots, \ell \), the number \( \xi_i \) occurs exactly \( k - \nu_i \) times in \( t \).

Then the sequence \( B_1, \ldots, B_n \) of B-splines of order \( k \) for the knot sequence \( t \) is a basis for \( P_{k,\xi,\nu} \), considered as functions on \( [t_k, t_{n+1}] \), i.e.,

\[
S_{k,t} = P_{k,\xi,\nu} \quad \text{on} \quad [t_k, t_{n+1}].
\]

**NOTE:** number of continuity conditions at \( \xi \) + number of knots at \( \xi \) = \( k \)

Proof. From the definition of divided differences, for any sufficiently smooth function \( g \), constants \( d_i, \ldots, d_{i+k} \) such that

\[
g[t_i, \ldots, t_{i+k}] = \sum_{r=i}^{i+k} d_r g^{(j_r)}(t_r),
\]

with \( j_r = \max\{s \mid r - s \geq i, t_{r-s} = t_r\} \), \( r = i, \ldots, i + k \). Thus

\[
B_i(x) = (t_{i+k} - t_i) \sum_{r=i}^{i+k} d_r (t_r - x)^{k-1-j_r}(k-1)/(k-1-j_r)!,
\]

which is clearly a piecewise polynomial function of order \( k \) with breakpoints at \( t_i, \ldots, t_{i+k} \), i.e., at some of the points \( \xi_2, \ldots, \xi_\ell \) (and possibly at some other points outside \( (\xi_1, \xi_{\ell+1}) \), but these don’t matter). \( B_i \) has a jump in its \( s \)-th derivative at the breakpoint \( \xi_j \) only if for some \( r \in [i, i+k] \), we have \( \xi_j = t_r \) and \( k - 1 - j_r = s \). Since \( j_r \) counts the number of \( t_m \)'s equal to \( t_r \) and with \( i \leq m < r \), it follows that \( j_r \) must be less than \( k - \nu_j \) which is the total number of \( t_m \)'s equal to \( \xi_j = t_r \) by construction of \( t \). This says that always \( s \geq \nu_j \), and so

\[
\text{jump}_{\xi_j} D^m B_i = 0 \quad \text{for} \quad m = 0, \ldots, \nu_j - 1.
\]

Therefore \( B_i \in P_{k,\xi,\nu} \), all \( i \).

Since there are \( n \) \( B_i \)'s and \( \dim P_{k,\xi,\nu} = n \), it suffices to show that the sequence \( (B_i)_{i=1}^n \) is linearly independent. This follows from:
Lemma (deBoor, Fix, 1973). Let \( \lambda_i \) be the linear functional given by the rule
\[
\lambda_i f = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi^{(k-1-r)}(\tau_i) D^r f(\tau_i),
\]
all \( f \), with \( \psi(t) = (t_{i+1} - t) \cdots (t_{i+k-1} - t)/(k-1)! \), and \( \tau_i \) some arbitrary point in the open interval \((t_i, t_{i+k})\). Then
\[
\lambda_i B_j = \delta_{ij}, \quad \text{all } j.
\]
Q. E. D.

B-SPLINE INTERPOLATION.

Let \( t = (t_i)_{1}^{n+k} \) be a nondecreasing knot sequence with \( t_i < t_{i+k}, \) all \( i \), and \( (B_i)_{1}^{n} \) the corresponding B-splines of order \( k \). The span \( S_{k,t} \) of \( B_1, \ldots, B_n \) is \( n \)-dimensional. Given a strictly increasing sequence \( \tau = (\tau_i)_{1}^{n} \) and function \( g \), the problem is to find \( f \in S_{k,t} \) such that \( f(\tau_i) = g(\tau_i) \) \( \forall i \). Or, find spline coefficients \( \alpha_j \) such that
\[
\sum_{j=1}^{n} \alpha_j B_j(\tau_i) = g(\tau_i), \quad i = 1, \ldots, n.
\]

Theorem (Schönberg, Whitney). The matrix \( (B_j(\tau_i)) \) is invertible \( \Leftrightarrow \)
\[
B_i(\tau_i) \neq 0, \quad i = 1, \ldots, n,
\]
i.e., \( t_i < \tau_i < t_{i+k}, \) all \( i \).

Theorem (Karlin). The matrix \( (B_j(\tau_i)) \) is totally positive (all minors \( \geq 0 \)).

Observation. \( (B_j(\tau_i)) \) has bandwidth less than \( k \) if \( t_i < \tau_i < t_{i+k}, \) all \( i \).
Spline (a.k.a. piecewise polynomial) representations.

Consider the case of a piecewise cubic polynomial \( g(x) \) with breakpoints \( \xi_1 < \xi_2 < \cdots < \xi_{n+1} \).

1. **Polynomial coefficients** \( c_{ji} \), \( 1 \leq i \leq n \), \( 1 \leq j \leq 4 \). On each interval \([\xi_i, \xi_{i+1})\),

\[
g(x) = P_i(x) = \sum_{j=1}^{4} c_{ji} (x - \xi_i)^{j-1},
\]

and by convention

\[
g(x) = \begin{cases} 
P_i(x), & x \leq \xi_1 \\
P_n(x), & x \geq \xi_{n+1}. 
\end{cases}
\]

2. **Truncated power basis**

\[
\phi_{ij} = \begin{cases} 
(x - \xi_i)^j / j!, & i = 1, \\
(x - \xi_i)^j / j!, & i = 2, \ldots, n, \\
0, & j = 0, \ldots, 3,
\end{cases}
\]

where

\[
(x - \xi_i)_+^j = \begin{cases} 
0, & x < \xi_i, \\
(x - \xi_i)^j, & x \geq \xi_i.
\end{cases}
\]

\[
g(x) = \sum_{1 \leq i \leq n, 0 \leq j \leq 3} \alpha_{ij} \phi_{ij}(x), \quad \xi_1 \leq x \leq \xi_{n+1}.
\]

Proof. For \( 1 \leq i \leq n \) and \( 0 \leq j \leq 3 \), \( \phi_{ij}(x) \in \mathcal{P}_{4,\xi} = \{ \text{piecewise cubic (order 4) polynomials with breakpoints } \xi_1, \ldots, \xi_{n+1} \} \). From (1), \( \dim \mathcal{P}_{4,\xi} = 4n \). The \( 4n \) vectors \( \phi_{ij}(x) \), \( 1 \leq i \leq n \), \( 0 \leq j \leq 3 \), are linearly independent. Therefore the \( \phi_{ij} \) are a basis for \( \mathcal{P}_{4,\xi} \), and any \( g \in \mathcal{P}_{4,\xi} \) is a unique linear combination of the \( \phi_{ij} \).

\[ \mathcal{P}_{k,\xi,\nu} = \{ \text{piecewise polynomials } g(x) \text{ of degree } \leq k - 1 \text{ with breakpoints } \xi_1 < \xi_2 < \cdots < \xi_{n+1} \text{ such that } g^{(\nu_i-1)}(x) \text{ exists at } \xi_i, \text{ and } 0 \leq \nu_i \leq k, \text{ for } 2 \leq i \leq n \} \].

The special case \( k = 4 \), \( \nu = (2, \ldots, 2) \) is called **piecewise Hermite cubics**. Let \( h_i = \xi_{i+1} - \xi_i \),

\[
c_0^i(x) = \frac{2}{h_i^3} (x - \xi_i)^2 \left( x - \xi_i + \frac{h_i}{2} \right), \quad \hat{c}_0^i(x) = \frac{1}{h_i^2} (x - \xi_i)(x - \xi_{i+1})^2,
\]

\[
c_1^i(x) = -\frac{2}{h_i^3} (x - \xi_i)^2 \left( x - \xi_{i+1} - \frac{h_i}{2} \right), \quad \hat{c}_1^i(x) = \frac{1}{h_i^2} (x - \xi_i)^2 (x - \xi_{i+1}).
\]

The \( C' \) piecewise Hermite cubics \( c_i(x), \hat{c}_i(x) \) are defined by

\[
c_1(x) = \begin{cases} 
0, & \text{on } [\xi_1, \xi_2], \\
\hat{c}_0^i(x), & \text{on } [\xi_2, \xi_{n+1}],
\end{cases} \quad \hat{c}_1(x) = \begin{cases} 
\hat{c}_0^0(x), & \text{on } [\xi_1, \xi_2] \\
0, & \text{on } [\xi_2, \xi_{n+1}],
\end{cases}
\]

for \( i = 2, \ldots, n, \)

\[
c_i(x) = \begin{cases} 
0, & \text{on } [\xi_1, \xi_{i-1}], \\
c_0^i(x), & \text{on } [\xi_{i-1}, \xi_i], \\
\hat{c}_0^i(x), & \text{on } [\xi_i, \xi_{i+1}], \\
0, & \text{on } [\xi_{i+1}, \xi_{n+1}],
\end{cases} \quad \hat{c}_i(x) = \begin{cases} 
0, & \text{on } [\xi_1, \xi_{i-1}] \\
\hat{c}_0^{i-1}(x), & \text{on } [\xi_{i-1}, \xi_i] \\
\hat{c}_0^i(x), & \text{on } [\xi_i, \xi_{i+1}] \\
0, & \text{on } [\xi_{i+1}, \xi_{n+1}],
\end{cases}
\]

\[
c_{n+1}(x) = \begin{cases} 
0, & \text{on } [\xi_1, \xi_n], \\
c_1^n(x), & \text{on } [\xi_n, \xi_{n+1}],
\end{cases} \quad \hat{c}_{n+1}(x) = \begin{cases} 
0, & \text{on } [\xi_1, \xi_n] \\
\hat{c}_1^n(x), & \text{on } [\xi_n, \xi_{n+1}].
\end{cases}
\]
\[ c_i(\xi_j) = \delta_{ij}, \quad c_i'(\xi_j) = 0, \quad 1 \leq i, j \leq n + 1. \]
\[ c_i(\xi_j) = \delta_{ij}, \quad \hat{c}_i(\xi_j) = 0, \quad 1 \leq i, j \leq n + 1. \]

(3) **Piecewise Hermite cubic basis** \( c_i(x), \hat{c}_i(x) \).

\[
g(x) = \sum_{i=1}^{n+1} y_i c_i(x) + d_i \hat{c}_i(x).
\]

Note that \( g(\xi_i) = y_i \) and \( g'(\xi_i) = d_i \).

Standard cubic splines correspond to the special case \( k = 4, \nu = (3, 3, \ldots, 3) \) of \( \mathcal{P}_{k,\xi,\nu} \). Basis functions \( B_i(x) \), called \( B \)-splines, for this space look like

The points \( t_i \) are called knots, which are not identical with the breakpoints \( \xi_i \). Precisely, given the breakpoint sequence \( \xi \), a standard choice for the knot sequence \( t \) is \( t_1 = t_2 = t_3 = t_4 = \xi_1 \), \( t_{i+3} = \xi_i \) for \( i = 2, \ldots, n \), \( t_{n+4} = t_{n+5} = t_{n+6} = t_{n+7} = \xi_{n+1} \). Then for any \( C^2 \) cubic spline \( g \in \mathcal{P}_{4,\xi,3} \), \( g(x) = \sum_{i=1}^{n+3} \alpha_i B_i(x) \), giving the

(4) **\( B \)-spline basis** \( B_i(x) \).

The real power of \( B \)-splines lies in

**Theorem.** For any \( k \geq 1 \), breakpoint sequence \( \xi_1 < \xi_2 < \cdots < \xi_{l+1} \), and smoothness sequence \( \nu = (\nu_i)_{l}^{1} \) with all \( \nu_i \leq k \), there exists a knot sequence \( t = (t_{i})_{1}^{n+k} \), where

\[
n = kl - \sum_{i=2}^{l} \nu_i,
\]

such that the \( B \)-splines \( B_1, \ldots, B_n \) of order \( k \) for the knot sequence \( t \) are a basis for \( \mathcal{P}_{k,\xi,\nu} \) on \( [t_k, t_{n+1}] \).
Cubic spline interpolants.

The $B$-spline representation $g(x) = \sum_{i=1}^{n+3} \alpha_i B_i(x)$ shows clearly there are $n + 3$ degrees of freedom. Thus fixing $g(\xi_i) = f(\xi_i)$ for $i = 1, \ldots, n+1$, leaves 2 degrees of freedom. Some standard choices are:

1. $g'(\xi_1) = f'(\xi_1)$, $g'(\xi_{n+1}) = f'(\xi_{n+1})$ : complete cubic spline;
2. $g''(\xi_1) = g''(\xi_{n+1}) = 0$ : natural cubic spline;
3. $g'''(x)$ is continuous at $\xi_2$ and $\xi_n$ : not-a-knot condition.

"The only good thing about a natural cubic spline is its name." (Carl deBoor).