CS4124

Answers to Selected Problems of HW #6

Problem 3.5.1

(a) Let $L_1 = \{a\}^+\{a^n b^n : n \in \mathbf{N}\}$. Then L_1 is the concatenation of two languages we know to be context-free and is itself context-free. Similarly $L_2 = \{a^n b^n : n \in \mathbf{N}\}\{b\}^+$ is also context-free. $\{a^m b^n : m \neq n\} = \{a^m b^n : m > n\} \cup \{a^m b^n : m < n\} = L_1 \cup L_2$ and is thus context-free.

(b) This language can be expressed as $\{a^n b^m : n \neq m\} \cup \Sigma^* a \Sigma^* b \Sigma^* a \Sigma^* \cup \Sigma^* b \Sigma^* a \Sigma^* b \Sigma^*$, all three of which are context-free, so that their union is also context-free.

(c) This language can be expressed as $L_1 \cup L_2 \cup L_3$, where

$$\begin{aligned} L_1 &= \{a^* b^n c^* d^n : n \geq 0\} \\ L_2 &= \{a^m b^* c^n d^* : m \leq n\} \\ L_2 &= \{a^m b^n c^p d^q : m + n = p + pq\}. \end{aligned}$$

Each of these languages can easily be recognized by a pushdown automaton created by modifying the one which accepts $\{a^n b^n : n \geq 0\}$. For L_1, we start from the automaton for $\{b^n d^n : n \geq 0\}$ and allow any number of as to be consumed at the beginning of the computation, and any number of cs between the bs and the ds, in each case without touching the stack. For L_2, we start from the automaton for $\{a^n c^n : n \geq 0\}$, apply the same technique of allowing arbitrary consumption of bs and ds (at the appropriate points in the computation), and also allow the machine to not pop an a from the stack in any circumstance when it could have done so (on reading a c). For L_3, we simply make the machine treat as and bs identically (except that the bs must follow the as), and similarly for cs and ds.

(d) This language is $\Sigma^* bb \Sigma^* \cup \Sigma^* \{ba^n ba^m b : n+1 \neq m\} \Sigma^* \cup a\Sigma^* \cup \Sigma^* a$. Each of these individual languages is context-free, so their union is also.

(e) This was shown directly by means of a grammar in problem 3.1.3 and by a pushdown automaton in problem 3.3.3.

Problem 3.5.14

(a) This language *is* context-free: it can be represented as $\{a^n b^n c^m : n, m \in \mathbf{N}\} \cup \{a^n b^m c^n : n, m \in \mathbf{N}\} \cup \{a^m b^n c^n : n, m \in \mathbf{N}\}$, each of which, being essentially the language $\{a^n b^n : n \in \mathbf{N}\}$ is context-free.

(b) This language *is* context-free: it can be represented as $\{a^n b^m c^p : n \neq m\} \cup \{a^n b^m c^p : n \neq p\} \cup \{a^n b^m c^p : m \neq p\}$, each of which, being essentially the language $\{a^n b^m : n \neq m\}$ is context-free.

(c) This language *is not* context-free: this language is the same language $\{a^n b^n c^n : n \in \mathbf{N}\}$ that was shown not to be context-free in example 3.5.2.

(d) This language *is* context-free. The language of strings over $\{a, b, c\}^*$ with different numbers of as and bs is context-free, as is the language with different numbers of bs and cs, and the language with different numbers of as and cs. The language in question is just their union, which is therefore context-free.

(e) This language is really just the set of all strings whose lengths are composite. It is not context-free for much the same reason that the language of strings of prime length is not context-free: the "gaps" represented by prime numbers do not follow a simple enough pattern to be context-free.

To see this in action, first apply the homomorphism $h(a) = h(b) = a$. The resulting language – which must be context-free if the original language was – is the language of strings of composite length over the alphabet $\{a\}$. If this language were context-free, it would also be regular, as any context-free language over an alphabet of a single symbol is regular (noted on page 147). And if this language were regular, its complement – the language of strings of prime length – would also be regular. But this language is not even context-free (problem 3.5.2a), so we have a contradiction to the assumption that the original language was context-free.

Problem 3.5.15

Since $L - R = L \cap \overline{R}$, it must be context-free, as the intersection of a context-free language (L) with a regular language ($\overline{R}$).

$R - L$ need not be context-free. If we restrict to the case $R = \Sigma^*$ (certainly a regular langauge!) then $R - L = \overline{L}$. Since we know that the class of context-free languages is not closed under complementation, it is not closed under the act of set difference from a regular set.

Problem 4.1.2

(a)

$$
\begin{aligned}
(q_0, \triangleright a\underline{b}b \sqcup bb \sqcup\sqcup\sqcup aba) &\vdash_M (q_0, \triangleright ab\underline{b} \sqcup bb \sqcup\sqcup\sqcup aba) \\
&\vdash_M (q_0, \triangleright abb\underline{\sqcup}bb \sqcup\sqcup\sqcup aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup \underline{b}b \sqcup\sqcup\sqcup aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup b\underline{b} \sqcup\sqcup\sqcup aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup bb\underline{\sqcup} \sqcup\sqcup aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup bb \sqcup \underline{\sqcup} \sqcup aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup bb \sqcup \sqcup\underline{\sqcup} aba) \\
&\vdash_M (q_0, \triangleright abb \sqcup bb \sqcup\sqcup\sqcup \underline{a}ba) \\
&\vdash_M (q_1, \triangleright abb \sqcup bb \sqcup \sqcup\underline{\sqcup} aba) \\
&\vdash_M (q_1, \triangleright abb \sqcup bb \sqcup \underline{\sqcup} \sqcup aba) \\
&\vdash_M (q_1, \triangleright abb \sqcup bb\underline{\sqcup} \sqcup\sqcup aba) \\
&\vdash_M (q_1, \triangleright abb \sqcup b\underline{b} \sqcup\sqcup\sqcup aba) \\
&\vdash_M (q_2, \triangleright abb \sqcup bb\underline{\sqcup} \sqcup\sqcup aba) \\
&\vdash_M (h, \triangleright abb \sqcup bb\underline{\sqcup} \sqcup\sqcup aba)
\end{aligned}
$$

(b) M scans right until it finds an a, then left until it finds a b, then right again until it finds a $\sqcup$, and then halts.

Problem 4.1.4

M moves the head to the left, replacing every other a with a $\sqcup$ (the first one to be replaced is the one immediately to the left of the a being scanned initially). If n is odd, then M finishes by looping forever on the $\sqcup$ to the right of the $\triangleright$; if n is even, then M halts on that blank.

Problem 4.1.6

(a+b) None of these are configurations; each is a quadruple, while we have defined Turing machine configurations to be triples.

(c)

i $(q, \triangleright ab, cd)$

ii $(q, \triangleright a, e)$

iii $(p, \triangleright aa \sqcup \sqcup, e)$

iv $(h, \triangleright \sqcup abc)$

Problem 4.1.7

$$\begin{aligned} K &= \{q_0, q_1, h\} \\ \Sigma &= \{\sqcup, \triangleright, a, b\} \\ s &= q_0 \\ H &= \{h\} \end{aligned}$$

δ is given by the following table:

q	σ	$\delta(q,\sigma)$
q_0	a	$(q_1, \rightarrow)$
q_0	b	$(q_0, \rightarrow)$
q_0	$\sqcup$	$(q_0, \rightarrow)$
q_0	$\triangleright$	$(q_0, \rightarrow)$
q_1	a	(h, a)
q_1	b	$(q_0, \rightarrow)$
q_1	$\sqcup$	$(q_0, \rightarrow)$
q_1	$\triangleright$	$(q_0, \rightarrow)$

Problem 4.1.8

(a)

$$\begin{aligned} K &= \{q_0, q_1, h\} \\ \Sigma &= \{\sqcup, \triangleright, a, b\} \\ s &= q_0 \\ H &= \{h\} \end{aligned}$$

δ is given by the following table ($\delta(q, \triangleright) = (q, \rightarrow)$ for all $q \in K$):

q	σ	$\delta(q,\sigma)$
q_0	a	$(q_1, \leftarrow)$
q_0	b	$(q_1, \leftarrow)$
q_0	$\sqcup$	$(q_1, \leftarrow)$
q_0	a	$(h, \leftarrow)$
q_0	b	$(h, \leftarrow)$
q_0	$\sqcup$	$(h, \leftarrow)$

(b)

$$\begin{aligned} K &= \{q_0\} \\ \Sigma &= \{\sqcup, \triangleright, a, b\} \\ s &= q_0 \\ H &= \{h\} \end{aligned}$$

δ is given by the following table ($\delta(q, \triangleright) = (q, \rightarrow)$ for all $q \in K$):

q	σ	$\delta(q,\sigma)$
q_0	a	$(q_0, \rightarrow)$
q_0	b	$(q_0, \rightarrow)$
q_0	$\sqcup$	$(q_0, \rightarrow)$

(c)

$$\begin{aligned} K &= \{q_0, q_1, h_0, h_1\} \\ \Sigma &= \{\sqcup, \triangleright, a, b\} \\ s &= q_0 \\ H &= \{h_0, h_1\} \end{aligned}$$

δ is given by the following table ($\delta(q, \triangleright) = (q, \rightarrow)$ for all $q \in K$):

q	σ	$\delta(q,\sigma)$
q_0	a	$(h_0, \leftarrow)$
q_0	b	$(h_0, \leftarrow)$
q_0	$\sqcup$	$(q_1, \leftarrow)$
q_1	a	$(h_1, \rightarrow)$
q_1	b	$(h_1, \rightarrow)$
q_1	$\sqcup$	$(h_1, \rightarrow)$

Problem 4.1.11

Because our programming schemata abstracts away individual states, we will list by tape contents the key configurations that this machine passes through:

$$
\begin{array}{lll}
(\triangleright\underline{\sqcup}aabb) & \vdash_M^* & (\triangleright\sqcup\underline{a}abb) \\
 & \vdash_M^* & (\triangleright\sqcup\underline{\sqcup}abb) \\
 & \vdash_M^* & (\triangleright\sqcup\sqcup abb\underline{\sqcup}\sqcup) \\
 & \vdash_M^* & (\triangleright\sqcup\sqcup abb\sqcup\underline{\sqcup}) \\
 & \vdash_M^* & (\triangleright\sqcup\sqcup abb\sqcup\underline{a}) \\
 & \vdash_M^* & (\triangleright\sqcup\underline{\sqcup}abb\sqcup a) \\
 & \vdash_M^* & (\triangleright\sqcup\sqcup abb\underline{\sqcup}a) \\
 & \vdash_M^* & (\triangleright\sqcup\underline{a}abb\sqcup a) \\
 & \vdash_M^* & (\triangleright\sqcup a\underline{a}bb\sqcup a) \\
 & \vdash_M^* & (\triangleright\sqcup a\underline{\sqcup}bb\sqcup a) \\
 & \vdash_M^* & (\triangleright\sqcup a\sqcup bb\underline{\sqcup}a) \\
 & \vdash_M^* & (\triangleright\sqcup a\sqcup bb\sqcup a\underline{\sqcup}) \\
 & \vdash_M^* & (\triangleright\sqcup a\sqcup bb\sqcup a\underline{a}) \\
 & \vdash_M^* & (\triangleright\sqcup a\underline{\sqcup}bb\sqcup aa) \\
 & \vdash_M^* & (\triangleright\sqcup a\underline{a}bb\sqcup aa) \\
 & \vdash_M^* & (\triangleright\sqcup aa\underline{b}b\sqcup aa) \\
 & \vdash_M^* & (\triangleright\sqcup aa\underline{\sqcup}b\sqcup aa) \\
 & \vdash_M^* & (\triangleright\sqcup aa\sqcup b\sqcup aa\underline{\sqcup}) \\
 & \vdash_M^* & (\triangleright\sqcup aa\sqcup b\sqcup aa\underline{b}) \\
 & \vdash_M^* & (\triangleright\sqcup aa\underline{\sqcup}b\sqcup aab) \\
 & \vdash_M^* & (\triangleright\sqcup aa\underline{b}b\sqcup aab) \\
 & \vdash_M^* & (\triangleright\sqcup aab\underline{b}\sqcup aab) \\
 & \vdash_M^* & (\triangleright\sqcup aab\underline{\sqcup}\sqcup aab) \\
 & \vdash_M^* & (\triangleright\sqcup aab\sqcup\sqcup aab\underline{\sqcup}) \\
 & \vdash_M^* & (\triangleright\sqcup aab\sqcup\sqcup aab\underline{b}) \\
 & \vdash_M^* & (\triangleright\sqcup aab\underline{\sqcup}\sqcup aabb) \\
 & \vdash_M^* & (\triangleright\sqcup aab\underline{b}\sqcup aabb) \\
 & \vdash_M^* & (\triangleright\sqcup aabb\underline{\sqcup}aabb) \\
 & \vdash_M^* & (\triangleright\sqcup aabb\sqcup aabb\underline{\sqcup})
\end{array}
$$

Note that this machine transforms $\underline{\sqcup}w\sqcup$ to $\sqcup w\sqcup w\underline{\sqcup}$ (not $\sqcup w\underline{\sqcup}w\sqcup$ as stated on page 190).