CS4124 Homework #1 Solution

Note: The definition of N in this course is: $N = \{0, 1, 2, \ldots\}$ other than $N = \{1, 2, \ldots\}$.

1.3.6
(a) R is a partial order. R is reflexive because every number is divisible by itself. R is anti-symmetric because if b is divisible by a(with $a \neq b$), then $a < b$, so it cannot be the case that a is divisible by b. R is transitive because if b is divisible by a then $b = na$, and similarly if c is divisible by b then $c = mb$. Then $c = nma$, so that c is divisible by a. R is not a total order — for many pairs of numbers a and b, it it not the case that neither a is divisible by b or that b is divisible by a(for example, $a = 2$ and $b = 3$).
(b) R is not a partial order, since it is not anti-symmetric. $((1, 2), (2, 1)) \in R$ and $((2, 1), (1, 2)) \in R$, but $(1, 2) \neq (2, 1)$. Since R is not a partial order, it is also not a total order.
(c) R is not a partial order, since it is not transitive. $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$. Since R is not a partial order, it is also not a total order.
(d) R is not a partial order, since it is not anti-symmetric.
(e) R is a partial order. R is not a total order.

1.4.2
(a) Let f from N to the odd natural numbers be given by $f(n) = 2n + 1$.
(b) Let f from the set of all integers to N be given by

$$f(x) = \begin{cases} 2x - 1 & x > 0 \\ -2x & x \leq 0 \end{cases}$$

(c) let $f : N \times N \times N \to N$ be given by

$$f(i, j, k) = \frac{1}{6}(i + j + k)^3 + \frac{1}{2}(i + j)^2 + i$$

1.5.2
We show this result by induction on n.
Basis step. For $n = 0$, we have that $n^4 - 4n^2 = 0$, which is divisible by 3.

Induction hypothesis. Assume that $n^4 - 4n^2 = 3r$ for some $r \in N$.
Induction step.

$$\begin{aligned}(n+1)^4 - 4(n+1)^2 &= n^4 + 4n^3 + 6n^2 + 4n + 1 - 4n^2 - 8n - 4 \\ &= n^4 + 4n^3 + 2n^2 - 4n - 3 \\ &= (n^4 - 4n^2) + 4n^3 + 6n^2 - 4n - 3\end{aligned}$$

Applying the inductive hypothesis, we can substitute $3r$ for $n^4 - 4n^2$ to get

$$\begin{aligned}(n^4 - 4n^2) + 4n^3 + 6n^2 - 4n - 3 &= 3r + 2n(2n^2 + 3n - 2) - 3 \\ &= 3r + 2n(2n-1)(n+2) - 3\end{aligned}$$

The first and third terms above are clearly divisible by 3; we need to show that the middle term is also divisible by 3. n must be of the form $3s$, $3s+1$, or $3s+2$. If $n = 3s$, then n is divisible by 3. If $n = 3s+1$, then $n+2 = 3s+3$, which is divisible by 3. And if $n = 3s+2$, then $2n-1 = 6s+4-1 = 6s+3$, which is divisible by 3. In each case, the middle term is divisible by 3.

1.5.9
Let S be uncountable and T be countable. Suppose $S - T$ were countable. Then S would be countable, because it is the union of the countable sets S and $S - T$. Thus $S - T$ cannot be countable.

1.5.x
Key idea:$\ldots 00017$ is not a integer.

1.8.2
(a) $(a \cup b)^*$
(b) $(a \cup b)^*$
(c) $(a \cup b)^*$
(d) $(a \cup b)^* a (a \cup b)^*$

1.8.5
(a) True.
(b) True.
(c) False.

(d) False.

1.8.6
(a) $\oslash \cup (ab \cup abc)^* ab$
(b) $(a \cup aa(ca \cup b)^* c)^*$
(c) $(c(b \cup a)^* b \cup c)^*$
(d) $(a \cup b)^*$
(e) $\oslash \cup ab(aab \cup b)^* a$

Solution for 1.3.2 and 1.7.2 is not included.