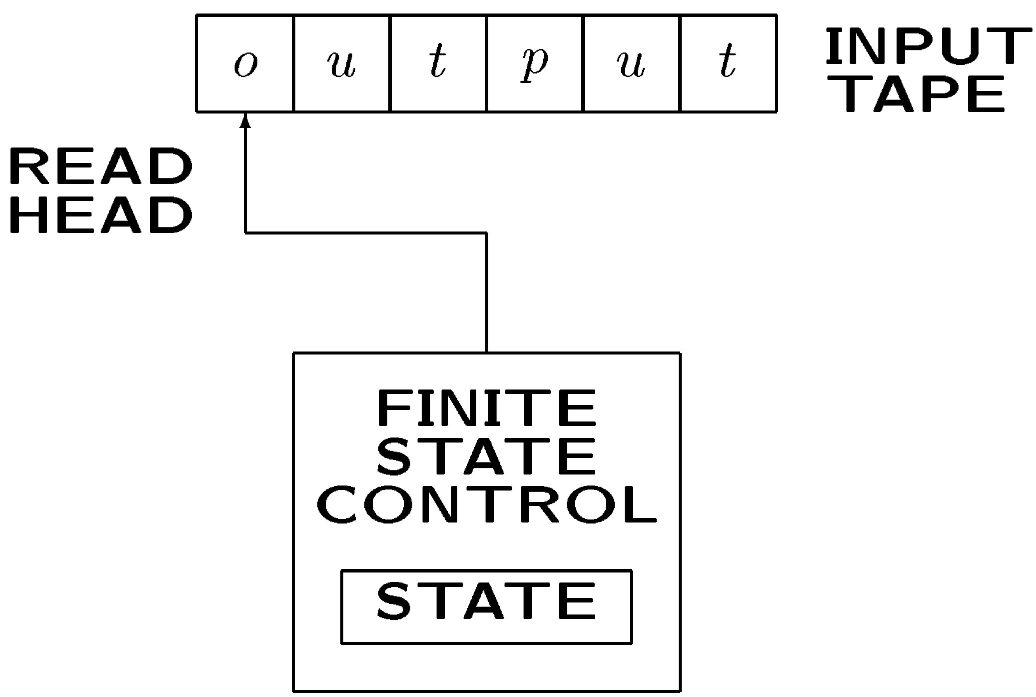
Formal Definition of DFA

A **deterministic finite automaton (DFA)** is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of **states;**
- Σ is the **input alphabet;**
- $\delta : Q \times \Sigma \to Q$ is the **transition function;**
- $q_0 \in Q$ is the **start state;** and
- $F \subset Q$ is the set of **final** or **accepting states.**

The Model

A typical mental model for a DFA looks like this:



Characteristics of The Model

- The symbols on the input tape are from the input alphabet Σ, one symbol per tape square.
- The read-only tape head examines one square at a time and proceeds only left to right. The computation ends when the tape head moves off the rightmost square.
- The **STATE** register contains the current state from Q. Initially contains q_0.
- The **FINITE STATE CONTROL** uses the transition function δ to implement computation steps.
- Acceptance depends on whether the **STATE** register contains a final state (from F) after the last step of the computation.

Example Using The Model

Let M_1 be the DFA given by:

$$\begin{aligned} M_1 &= (Q_1, \Sigma, \delta_1, q_0, F_1) \\ Q_1 &= \{q_0, q_1, q_2, q_3\} \\ \Sigma &= \{a, b\} \\ F_1 &= \{q_3\} \end{aligned}$$

The transition function $\delta_1 : Q_1 \times \Sigma \to Q_1$ is given by this table:

q	$\delta_1(q, a)$	$\delta_1(q, b)$
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_3	q_0
q_3	q_2	q_1

Try the model on these inputs: $abbab$ and $baabba$.

Acceptance

Informally, a string $w \in \Sigma^*$ is **accepted** by M if M is in a final state after reading w.

To define acceptance formally, we need the notions of **configuration** and of the **yields relation.**

Configurations

A **configuration** of M is an element of $Q \times \Sigma^*$.

For an input $w \in \Sigma^*$, the **initial** or **start configuration** is

$$(q_0, w).$$

The **yields (in one step) relation** $\underset{M}{\vdash}$ is a binary relation on $Q \times \Sigma^*$. If $q_i, q_j \in Q$, $\sigma \in \Sigma$ and $v \in \Sigma^*$, then

$$(q_i, \sigma v) \ \underset{M}{\vdash} \ (q_j, v)$$

if and only if

$$\delta(q_i, \sigma) \ = \ q_j.$$

Example

For the previous example M_1 and the input $w = abbab$, the initial configuration is

$$(q_0, abbab).$$

The initial configuration yields what configuration in one step:

$$(q_0, abbab) \underset{M_1}{\vdash} \boxed{\quad ?}$$

How does the computation go when expressed using configurations and the yields relation $\boxed{\quad ?}$

Yields in t Steps

A recursive definition for the **yields in t steps relation** $\overset{t}{\underset{M}{\vdash}}$ is

- **Basis.** For all $q_i \in Q$ and all $v \in \Sigma^*$,

$$(q_i, v) \overset{0}{\underset{M}{\vdash}} (q_i, v).$$

- **Recursive Step.** If

$$(q_i, x) \overset{t}{\underset{M}{\vdash}} (q_j, y).$$

and

$$(q_j, y) \underset{M}{\vdash} (q_k, z).$$

then

$$(q_i, x) \overset{t+1}{\underset{M}{\vdash}} (q_k, z).$$

Yields in Zero or More Steps

The **yields (in zero or more steps) relation** $\overset{*}{\underset{M}{\vdash}}$ is the union

$$\overset{*}{\underset{M}{\vdash}} \;=\; \bigcup_{t \geq 0} \overset{t}{\underset{M}{\vdash}}.$$

In other words,

$$(q_i, x) \;\; \overset{*}{\underset{M}{\vdash}} \;\; (q_j, y)$$

holds if and only if there exists a $t \geq 0$ such that

$$(q_i, x) \;\; \overset{t}{\underset{M}{\vdash}} \;\; (q_j, y).$$

Exercise

For the previous example M_1, give all the configurations (q_k, z) satisfying

$$(q_2, baab) \ \overset{*}{\underset{M_1}{\vdash}} \ (q_k, z).$$

Acceptance Again

A string $w \in \Sigma^*$ is **accepted** by the DFA M if

$$(q_0, w) \ \overset{*}{\underset{M}{\vdash}} \ (q_i, \lambda),$$

where $q_i \in F$.

The **language** $L(M)$ **accepted** by the DFA M is the set of all strings accepted by M.

Said another way,

$$L(M) \ = \ \left\{ w \in \Sigma^* \mid (q_0, w) \overset{*}{\underset{M}{\vdash}} (q_i, \lambda) \text{ and } q_i \in F \right\}.$$

Exercises

Suppose $M_2 = (Q_2, \Sigma, \delta_2, q_0, F_2)$ has transition function δ_2 given by

q	$\delta_2(q,a)$	$\delta_2(q,b)$
q_0	q_0	q_1
q_1	q_0	q_0

and that $F_2 = \{q_0\}$. What is $L(M_2)$?

Said another way, what is

$$L(M_2) \;=\; \left\{ w \in \{a,b\}^* \mid (q_0, w) \overset{*}{\underset{M_2}{\vdash}} (q_0, \lambda) \right\}?$$

What is $L(M_1)$?

Said another way, what is

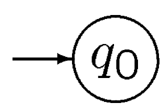
$$L(M_1) \;=\; \left\{ w \in \{a,b\}^* \mid (q_0, w) \overset{*}{\underset{M_1}{\vdash}} (q_3, \lambda) \right\}?$$

State Diagrams

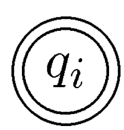
A DFA $M = (Q, \Sigma, \delta, q_0, F)$ has a graph representation called a **state diagram.**

The **state diagram** G for M has

- Node set Q.
- An arc from q_i to q_j labeled σ if $\delta(q_i, \sigma) = q_j$.
- The start state is designated

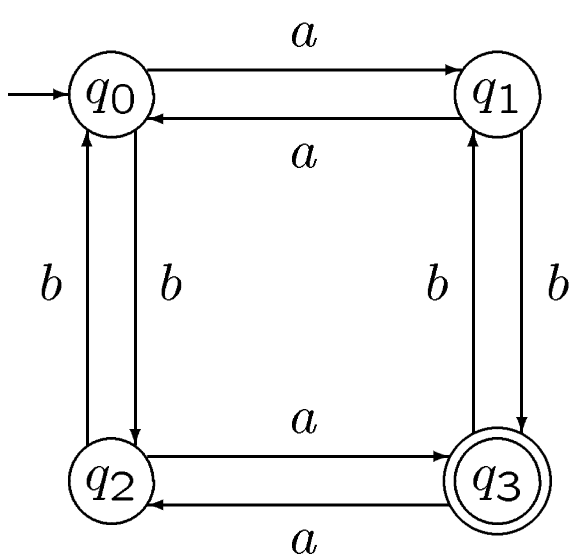


- A final state q_i is designated



Example

The state diagram for M_1 is



- Every node has $|\Sigma|$ outgoing arcs, one for each symbol in Σ.

- A string $w \in \Sigma^*$ determines a path in G from q_0 to the last state in the computation on input w.

- Follow the path for $w = abaab$.

Extended Transition Function

The **extended transition function**

$$\widehat{\delta} : Q \times \Sigma^* \to Q$$

is defined recursively as follows.

- **Basis:** If $|w| = 0$, then

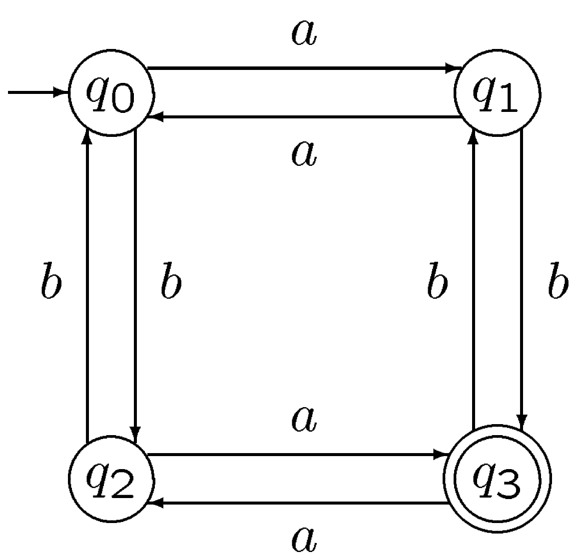
$$\widehat{\delta}(q_i, w) = q_i.$$

- **Recursive Step:** If $|w| > 0$, then $w = u\sigma$, where $u \in \Sigma^*$ and $\sigma \in \Sigma$. Define

$$\widehat{\delta}(q_i, w) \;=\; \delta(\widehat{\delta}(q_i, u), \sigma).$$

Example

Start with M_1 again:



Use the definition of the extended transition function to compute

$$\hat{\delta}(q_0, ba) = \boxed{\quad ?}$$

$$\hat{\delta}(q_2, abab) = \boxed{\quad ?}$$

Alternate Definition of Acceptance

A string $w \in \Sigma^*$ is **accepted** by the DFA M if

$$\widehat{\delta}(q_0, w) \in F.$$

The **language** $L(M)$ **accepted** by the DFA M is

$$L(M) = \left\{ w \in \Sigma^* \mid \widehat{\delta}(q_0, w) \in F \right\}.$$

Example

Recall $M_2 = (\{q_0, q_1\}, \{a, b\}, \delta_2, q_0, \{q_0\})$ with transition function δ_2 given by

q	$\delta_2(q, a)$	$\delta_2(q, b)$
q_0	q_0	q_1
q_1	q_0	q_0

By definition,

$$L(M_2) \;=\; \left\{ w \in \{a, b\}^* \mid \widehat{\delta}(q_0, w) = q_0 \right\}.$$

From the observations

$$\begin{aligned} \widehat{\delta}(q_i, wa) &= q_0 \\ \widehat{\delta}(q_i, wb) &= q_{1-i} \\ \widehat{\delta}(q_i, wbb) &= q_i, \end{aligned}$$

we conclude that

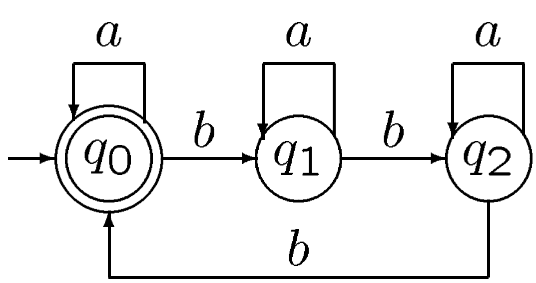
$$\begin{aligned} \Sigma^*\{a\} &\subset L(M_2) \\ L(M_2)\{bb\} &\subset L(M_2) \\ w \in L(M_2) &\iff wb \notin L(M_2). \end{aligned}$$

Constructing DFAs

Construct a DFA M_3 that accepts

$$L_3 = \{x \in \{a,b\}^* \mid n_b(x) \equiv 0 \bmod 3\}.$$

Need three states, to remember the number of b's modulo 3.



Construct a DFA M_5 that accepts

$$L_5 = \{x \in \{a,b\}^* \mid n_b(x) \equiv 0 \bmod 5\}$$

? (Use states p_0, p_1, p_2, p_3, p_4.)

Union of DFA Languages

EXERCISE.

Construct a DFA $M_{3,5}$ that accepts

$$\begin{aligned} L_{3,5} &= L_3 \cup L_5 \\ &= \{x \in \{a,b\}^* \mid n_b(x) \equiv 0 \bmod 3 \text{ or} \\ &\qquad n_b(x) \equiv 0 \bmod 5\} \end{aligned}$$

?

Union of DFA Languages

Now generalize the previous construction.

Theorem. If $L_1 \subset \Sigma^*$ is accepted by a DFA $M_1 = (Q_1, \Sigma, \delta_1, q_0, F_1)$ and $L_2 \subset \Sigma^*$ is accepted by a DFA $M_2 = (Q_2, \Sigma, \delta_2, p_0, F_2)$, then there is a DFA that accepts $L_1 \cup L_2$.

Proof: Define the DFA

$$M = (Q_1 \times Q_2, \Sigma, \delta', (q_0, p_0), F'),$$

where

$$\begin{aligned} F' &= (F_1 \times Q_2) \cup (Q_1 \times F_2) \\ \delta'((q,p),\sigma) &= (\delta_1(q,\sigma), \delta_2(p,\sigma)). \end{aligned}$$

Intuitively, M runs M_1 and M_2 in parallel on the same input.

Use the equation

$$L(M) = \left\{ w \in \Sigma^* \mid \hat{\delta}'((q_0, p_0), w) \in F' \right\}$$

to show that

$$L(M) = L_1 \cup L_2.$$

Partitioning

Start with a DFA $M = (Q, \Sigma, \delta, q_0, F)$.

Suppose that

$$Q \;=\; \{q_0, q_1, \ldots, q_{n-1}\}.$$

For each state q_i, define the associated language

$$L_{q_i} \;=\; \{w \in \Sigma^* \mid \widehat{\delta}(q_0, w) = q_i\}.$$

Then

$$L_{q_0}, L_{q_1}, \ldots, L_{q_{n-1}}$$

is a partition of Σ^*.

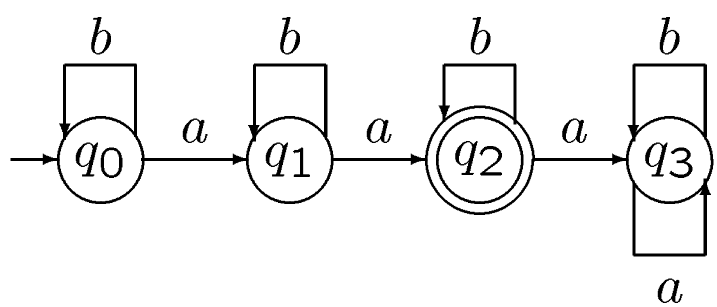
Why?

Example

Consider the language

$$L_6 = \{w \in \{a, b\}^* \mid n_a(w) = 2\}.$$

One DFA M_6 that accepts L_6 is given by



State q_3 is a **dead** or **error** state.

Find the partition:

L_{q_0} = ? L_{q_1} = ?

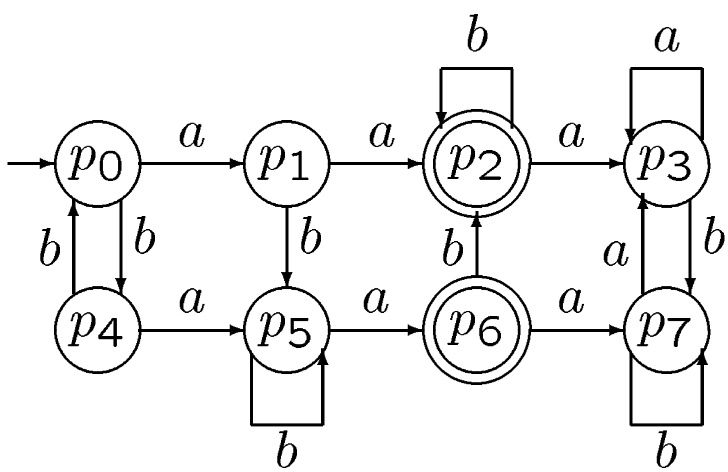
L_{q_2} = ? L_{q_3} = ?

Observe that

$$L_6 = L_{q_2}.$$

Example

Another DFA M_6' that accepts L_6 is given by



Again, find the partition ?

Express L_6 using the partition ?

Complement of Languages

Theorem. For every DFA $M = (Q, \Sigma, \delta, q_0, F)$, there is a DFA that accepts the complement $\Sigma^* - L(M)$.

Proof: The following DFA works:

$$M' \;=\; (Q, \Sigma, \delta, q_0, Q - F).$$

Intuitively, M' computes exactly as M does, but rejects when M accepts and accepts when M rejects.

Formally, we have that

$$\begin{aligned}
L(M') &= \left\{ w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in Q - F \right\} \\
&= \bigcup_{q_i \in Q - F} L_{q_i} \\
&= \Sigma^* - \bigcup_{q_i \in F} L_{q_i} \\
&= \Sigma^* - \left\{ w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F \right\} \\
&= \Sigma^* - L(M).
\end{aligned}$$

Exercise

Give a DFA that accepts the following language:

$$L_8 = \{w \in \{a, b\}^* \mid n_a(w) \neq 1 \text{ and } n_b(w) \neq 1\}.$$

What is the partition corresponding to your DFA ?

Express L_8 using the partition.