CS 4104: Data and Algorithm Analysis

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Parallel Algorithms

- **Running time**: $T(n, p)$ where n is the problem size, p is number of processors.
- **Speedup**: $S(p) = T(n, 1)/T(n, p)$.
  - A comparison of the time for a (good) sequential algorithm vs. the parallel algorithm in question.
- Problem: Best sequential algorithm may not be the same as the best algorithm for p processors, which may not be the best for ∞ processors.
- Efficiency: $E(n, p) = S(p)/p = T(n, 1)/(pT(n, p))$.
- Ratio of the time taken for 1 processor vs. the total time required for p processors.
  - Measure of how much the p processors are used (not wasted).
  - Optimal efficiency $= 1 =$ speedup by factor of p.

Parallel Algorithm Design

Approach (1): Pick p and write best algorithm.

- Would need a new algorithm for every p!

Approach (2): Pick best algorithm for $p = \infty$, then convert to run on p processors.

Hopefully, if $T(n, p) = X$, then $T(n, p/k) \approx kX$ for $k > 1$.

Using one processor to **emulate** k processors is called the **parallelism folding principle**.

Parallel Algorithm Design (2)

Some algorithms are only good for a large number of processors.

$$\begin{aligned} T(n, 1) &= n \\ T(n, n) &= \log n \\ S(n) &= n/\log n \\ E(n, n) &= 1/\log n \end{aligned}$$

For $p = 256$, $n = 1024$.
$T(1024, 256) = 4 \log 1024 = 40$.
For $p = 16$, running time $= 1024/16 * \log 1024 = 640$.
Speedup < 2, efficiency $= 1024/(16 * 640) = 1/10$.

Amdahl's Law

Think of an algorithm as having a **parallelizable** section and a **serial** section.

Example: 100 operations.

- 80 can be done in parallel, 20 must be done in sequence.

Then, the best speedup possible leaves the 20 in sequence, or a speedup of $100/20 = 5$.

Amdahl's law:

$$\begin{aligned}\text{Speedup} &= (\mathcal{S}+\mathcal{P})/(\mathcal{S}+\mathcal{P}/N) \\ &= 1/(\mathcal{S}+\mathcal{P}/N) \leq 1/\mathcal{S},\end{aligned}$$

for $\mathcal{S} =$ serial fraction, $\mathcal{P} =$ parallel fraction, $\mathcal{S}+\mathcal{P} = 1$.

Amdahl's Law Revisited

However, this version of Amdahl's law applies to a fixed problem size.

What happens as the problem size grows?
Hopefully, $\mathcal{S} = f(n)$ with $\mathcal{S}$ shrinking as n grows.

Instead of fixing problem size, fix execution time for increasing number N processors (and thus, increasing problem size).

$$\begin{aligned}
\text{Scaled Speedup} &= (\mathcal{S} + \mathcal{P} \times N)/(\mathcal{S} + \mathcal{P}) \\
&= \mathcal{S} + \mathcal{P} \times N \\
&= \mathcal{S} + (1 - \mathcal{S}) \times N \\
&= N + (1 - N) \times \mathcal{S}.
\end{aligned}$$

Models of Parallel Computation

Single Instruction Multiple Data (SIMD)

- All processors operate the same instruction in step.
- Example: Vector processor.

Pipelined Processing:

- Stream of data items, each pushed through the same sequence of several steps.

Multiple Instruction Multiple Data (MIMD)

- Processors are independent.

MIMD Communications (1)

Interconnection network:

- Each processor is connected to a limited number of neighbors.
- Can be modeled as (undirected) graph.
- Examples: Array, mesh, N-cube.
- It is possible for the cost of communications to dominate the algorithm (and in fact to limit parallelism).
- **Diameter**: Maximum over all pairwise distances between processors.
- Tradeoff between diameter and number of connections.

MIMD Communications (2)

Shared memory:

- Random access to global memory such that any processor can access any variable with unit cost.
- In practice, this limits number of processors.
- Exclusive Read/Exclusive Write (EREW).
- Concurrent Read/Exclusive Write (CREW).
- Concurrent Read/Concurrent Write (CRCW).

Addition

Problem: Find the sum of two n-bit binary numbers.

Sequential Algorithm:

- Start at the low end, add two bits.
- If necessary, carry bit is brought forward.
- Can't do ith step until $i - 1$ is complete due to uncertainty of carry bit (?).

Induction: (Going from $n - 1$ to n implies a sequential algorithm)

Parallel Addition

Divide and conquer to the rescue:

- Do the sum for top and bottom halves.
- What about the carry bit?

Strengthen induction hypothesis:

- Find the sum of the two numbers **with** or **without** the carry bit.

After solving for $n/2$, we have L, L_c, R, and R_c.

Can combine pieces in constant time.

Parallel Addition (2)

The $n/2$-size problems are independent.
Given enough processors,

$$T(n, n) = T(n/2, n/2) + O(1) = O(\log n).$$

We need only the EREW memory model.

Maximum-finding Algorithm: EREW

"Tournament" algorithm:

- Compare pairs of numbers, the "winner" advances to the next level.
- Initially, have $n/2$ pairs, so need $n/2$ processors.
- Running time is $O(\log n)$.

That is faster than the sequential algorithm, but what about efficiency?

$$E(n, n/2) \approx 1/\log n.$$

Why is the efficiency so low?

More Efficient EREW Algorithm

Divide the input into $n/\log n$ groups each with $\log n$ items.

Assign a group to each of $n/\log n$ processors.

Each processor finds the maximum (sequentially) in $\log n$ steps.

Now we have $n/\log n$ "winners".

Finish tournament algorithm.
$T(n, n/\log n) = O(\log n)$.
$E(n, n/\log n) = O(1)$.

More Efficient EREW Algorithm (2)

But what could we do with more processors?
A parallel algorithm is **static** if the assignment of processors to actions is predefined.

- We know in advance, for each step i of the algorithm and for each processor p_j, the operation and operands p_j uses at step i.

This maximum-finding algorithm is static.

- All comparisons are pre-arranged.

Brent's Lemma

Lemma 12.1: If there exists an EREW static algorithm with $T(n,p) \in O(t)$, such that the total number of steps (over all processors) is s, then there exists an EREW static algorithm with $T(n, s/t) \in O(t)$.

Proof:

- Let $a_i, 1 \leq i \leq t$, be the total number of steps performed by all processors in step i of the algorithm.
- $\sum_{i=1}^{t} a_i = s$.
- If $a_i \leq s/t$, then there are enough processors to perform this step without change.
- Otherwise, replace step i with $\lceil a_i/(s/t) \rceil$ steps, where the s/t processors emulate the steps taken by the original p processors.

Brent's Lemma (2)

- The total number of steps is now

$$\begin{aligned}\sum_{i=1}^{t} \lceil a_i/(s/t) \rceil &\leq \sum_{i=1}^{t} (a_i t/s + 1) \\ &= t + (t/s)\sum_{i=1}^{t} a_i = 2t.\end{aligned}$$

Thus, the running time is still $O(t)$.

Intuition: You have to split the s work steps across the t time steps somehow; things can't **always** be bad!

Maximum-finding: CRCW

- Allow concurrent writes to a variable only when each processor writes the same thing.
- Associate each element x_i with a variable v_i, initially "1".
- For each of $n(n-1)/2$ processors, processor p_{ij} compares elements i and j.
- First step: Each processor writes "0" to the v variable of the smaller element.
  - Now, only one v is "1".
- Second step: Look at all $v_i, 1 \leq i \leq n$.
  - The processor assigned to the max element writes that value to MAX.

Efficiency of this algorithm is **very** poor!

- "Divide and crush."

Maximum-finding: CRCW (2)

More efficient (but slower) algorithm:

- Given: n processors.
- Find maximum for each of $n/2$ pairs in constant time.
- Find max for $n/8$ groups of 4 elements (using 8 proc/group) each in constant time.
- Square the group size each time.
- Total time: $O(\log \log n)$.

Parallel Prefix

- Let $\cdot$ be any associative binary operation.
  - Ex: Addition, multiplication, minimum.
- Problem: Compute $x_1 \cdot x_2 \cdot \ldots \cdot x_k$ for all $k, 1 \leq k \leq n$.
- Define $\mathrm{PR}(i, j) = x_i \cdot x_{i+1} \cdot \ldots \cdot x_j$.
  We want to compute $\mathrm{PR}(1, k)$ for $1 \leq k \leq n$.
- Sequential alg: Compute each prefix in order
  - $O(n)$ time required (using previous prefix)
- Approach: Divide and Conquer
  - IH: We know how to solve for $n/2$ elements.

1. $\mathrm{PR}(1, k)$ and $\mathrm{PR}(n/2 + 1, n/2 + k)$ for $1 \leq k \leq n/2$.
2. $\mathrm{PR}(1, m)$ for $n/2 < m \leq n$ comes from $\mathrm{PR}(1, n/2) \cdot \mathrm{PR}(n/2 + 1, m)$ – from IH.

Parallel Prefix (2)

- **Complexity**: (2) requires $n/2$ processors and CREW for parallelism (all read middle position).
- $T(n, n) = O(\log n); \quad E(n, n) = O(1/\log n)$.
  Brent's lemma no help: $O(n \log n)$ total steps.

Better Parallel Prefix

- E is the set of all x_is with i even.
- If we know $\mathrm{PR}(1, 2\mathrm{i})$ for $1 \leq i \leq n/2$ then $\mathrm{PR}(1, 2\mathrm{i}+1) = \mathrm{PR}(1, 2\mathrm{i}) \cdot \mathrm{x}_{2\mathrm{i}+1}$.
- Algorithm:
  - Compute in parallel $x_{2i} = x_{2i-1} \cdot x_{2i}$ for $1 \leq i \leq n/2$.
  - Solve for E (by induction).
  - Compute in parallel $x_{2i+1} = x_{2i} \cdot x_{2i+1}$.
- Complexity:
  $T(n,n) = O(\log n)$. $S(n) = S(n/2) + n - 1$, so $S(n) = O(n)$.
  for $S(n)$ the total number of steps required to process n elements.
- So, by Brent's Lemma, we can use $O(n/\log n)$ processors for $O(1)$ efficiency.

Routing on a Hypercube

Goal: Each processor P_i simultaneously sends a message to processor $P_{\sigma(i)}$ such that no processor is the destination for more than one message.

Problem:

- In an n-cube, each processor is connected to n other processors.
- At the same time, each processor can send (or receive) only one message per time step on a given connection.
- So, two messages cannot use the same edge at the same time – one must wait.

Randomizing Switching Algorithm

It can be shown that any deterministic algorithm is $\Omega(2^{n^a})$ for some $a > 0$, where 2^n is the number of messages.

A node i (and its corresponding message) has binary representation $i_1 i_2 \cdots i_n$.

Randomization approach:

(a) Route each message from i to j to a random processor r (by a randomly selected route).

(b) Continue the message from r to j by the shortest route.

Randomized Switching (2)

Phase (a):

```
for (each message at i)
cobegin
  for (k = 1 to n)
    T[i, k] = RANDOM(0, 1);
  for (k = 1 to n)
    if (T[i, k] = 1)
      Transmit i along dimension k;
coend;
```

Randomized Switching (3)

Phase (b):

```
for (each message i)
cobegin
  for (k = 1 to n)
    T[i, k] =
       Current[i, k] EXCLUSIVE_OR Dest[i, k];
  for (k = 1 to n)
    if (T[i, k] = 1)
      Transmit i along dimension k;
coend;
```

Randomized Switching (4)

With high probability, each phase completes in $O(\log n)$ time.

- It is possible to get a really bad random routing, but this is unlikely.
- In contrast, it is very possible for any correlated group of messages to generate a bottleneck.

Sorting on an array

Given: n processors labeled $P_1, P_2, \cdots, P_n$ with processor P_i initially holding input x_i.

P_i is connected to P_{i-1} and P_{i+1} (except for P_1 and P_n).

- Comparisons/exchanges possible only for adjacent elements.

```
Algorithm ArraySort(X, n) {
  do in parallel ceil(n/2) times {
    Exchange-compare(P[2i-1], P[2i]); // Odd
    Exchange-compare(P[2i], P[2i+1]); // Even
  }
}
```

A simple algorithm, but will it work?

Parallel Array Sort

7	3	6	5	**8**	1	4	2
3	7	5	6	1	**8**	2	4
3	5	7	1	6	2	**8**	4
3	5	1	7	2	6	4	**8**
3	1	5	2	7	4	6	**8**
1	3	2	5	4	7	6	**8**
1	2	3	4	5	6	7	**8**
1	2	3	4	5	6	7	**8**
1	2	3	4	5	6	7	**8**

Correctness of Odd-Even Transpose

Theorem 12.2: When Algorithm ArraySort terminates, the numbers are sorted.

Proof: By induction on n.

Base Case: 1 or 2 elements are sorted with one comparison/exchange.

Induction Step:

- Consider the maximum element, say x_m.
- Assume m odd (if even, it just won't exchange on first step).
- This element will move one step to the right each step until it reaches the rightmost position.

Correctness (2)

- The position of x_m follows a diagonal in the array of element positions at each step.
- Remove this diagonal, moving comparisons in the upper triangle one step closer.
- The first row is the nth step; the right column holds the greatest value; the rest is an $n - 1$ element sort (by induction).

Sorting Networks

When designing parallel algorithms, need to make the steps independent.

Ex: Mergesort split step can be done in parallel, but the join step is nearly serial.

- To parallelize mergesort, we must parallelize the merge.

Batcher's Algorithm

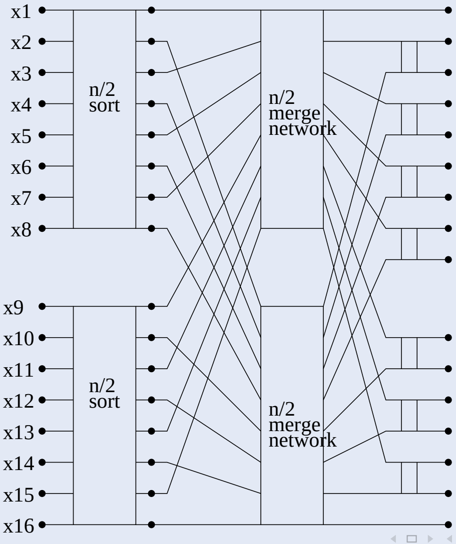
For n a power of 2, assume $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are sorted sequences.

Let $x_1, x_2, \cdots, x_{2n}$ be the final merged order.

Need to merge disjoint parts of these sequences in parallel.

- Split a, b into odd- and even- index elements.
- Merge a_{odd} with b_{odd}, a_{even} with b_{even}, yielding $o_1, o_2, \cdots, o_n$ and $e_1, e_2, \cdots, e_n$ respectively.

Batcher's Sort Image



Batcher's Algorithm Correctness

Theorem 12.3: For all i such that $1 \leq i \leq n - 1$, we have $x_{2i} = \min(o_{i+1}, e_i)$ and $x_{2i+1} = \max(o_{i+1}, e_i)$.

Proof:

- Since e_i is the ith element in the sorted even sequence, it is $\geq$ at least i even elements.
- For each even element, e_i is also $\geq$ an odd element.
- So, $e_i \geq 2i$ elements, or $e_i \geq x_{2i}$.
- In the same way, $o_{i+1} \geq i + 1$ odd elements, $\geq$ at least $2i$ elements all together.
- So, $o_{i+1} \geq x_{2i}$.
- By the pigeonhole principle, e_i and o_{i+1} must be x_{2i} and x_{2i+1} (in either order).

Batcher Sort Complexity

- Total number of comparisons for merge:

$$T_M(2n) = 2T_M(n) + n - 1; \quad T_M(1) = 1.$$

  Total number of comparisons is $O(n \log n)$, but the depth of recursion (parallel steps) is $O(\log n)$.
- Total number of comparisons for the sort is:

$$T_S(2n) = 2T_S(n) + O(n \log n), \quad T_S(2) = 1.$$

  So, $T_S(n) = O(n \log^2 n)$.
- The circuit requires n processors in each column, with depth $O(\log^2 n)$, for a total of $O(n \log^2 n)$ processors and $O(\log^2 n)$ time.
- The processors only need to do comparisons with two inputs and two outputs.

Matrix-Vector Multiplication

Problem: Find the product $x = A\mathbf{b}$ of an m by n matrix A with a column vector $\mathbf{b}$ of size n.

Systolic solution:

- Use n processor elements arranged in an array, with processor P_i initially containing element b_i.
- Each processor takes a partial computation from its left neighbor and a new element of A from above, generating a partial computation for its right neighbor.