Divide and Conquer Algorithms

T. M. Murali

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Divide and Conquer

- Break up a problem into several parts.
- Solve each part recursively.
- Solve base cases by brute force.
- Efficiently combine solutions for sub-problems into final solution.

Common use:
- Partition problem into two equal sub-problems of size $n/2$.
- Solve each part recursively.
- Combine the two solutions in $O(n)$ time.
- Resulting running time is $O(n \log n)$.
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Mergesort

Sort

**INSTANCE:** Nonempty list $L = x_1, x_2, \ldots, x_n$ of integers.

**SOLUTION:** A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

- Mergesort is a divide-and-conquer algorithm for sorting.
  1. Partition $L$ into two lists $A$ and $B$ of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively.
  2. Recursively sort $A$.
  3. Recursively sort $B$.
  4. Merge the sorted lists $A$ and $B$ into a single sorted list.
Merging Two Sorted Lists

- Merge two sorted lists $A = a_1, a_2, \ldots, a_k$ and $B = b_1, b_2, \ldots, b_l$.
  - Maintain a *current* pointer for each list.
  - Initialise each pointer to the front of the list.
  - While both lists are nonempty:
    - Let $a_i$ and $b_j$ be the elements pointed to by the *current* pointers.
    - Append the smaller of the two to the output list.
    - Advance the current pointer in the list that the smaller element belonged to.
  - EndWhile
  - Append the rest of the non-empty list to the output.

Running time of this algorithm is $O(k + l)$. 

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Analysing Mergesort

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Worst-case running time for $n$ elements ≤ Worst-case running time for $\lfloor n/2 \rfloor$ elements + Worst-case running time for $\lceil n/2 \rceil$ elements + Time to split the input into two lists + Time to merge two sorted lists.

Assume $n$ is a power of 2.

Define $T(n) \equiv$ Worst-case running time for $n$ elements, for every $n \geq 1$.

$T(n) \leq 2T(n/2) + cn$, $n > 2$

$T(2) \leq c$

Three basic ways of solving this recurrence relation:
1. “Unroll” the recurrence (somewhat informal method).
2. Guess a solution and substitute into recurrence to check.
3. Guess solution in $O()$ form and substitute into recurrence to determine the constants.
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T(n) \leq 2T(n/2) + cn, \quad n > 2
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Unrolling the recurrence

Recursion tree has $\log n$ levels. Total work done at each level is $cn$. Running time of the algorithm is $cn \log n$.

Use this method only to get an idea of the solution.

Figure 5.1 Unrolling the recurrence $T(n) \leq 2T(n/2) + O(n)$. 
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Figure 5.1 Unrolling the recurrence $T(n) \leq 2T(n/2) + O(n)$. 
Substituting a Solution into the Recurrence

- Guess that the solution is $T(n) \leq cn \log n$ (logarithm to the base 2).
- Use induction to check if the solution satisfies the recurrence relation.

Base case: $n = 2$. Is $T(2) = c \leq 2c \log 2$? Yes.

(Strong) Inductive hypothesis: assume $T(m) \leq cm \log_2 m$ for all $m < n$.

Therefore, $T(n/2) \leq (cn/2) \log(n/2)$.

Inductive step: Prove $T(n) \leq cn \log n$.

$T(n) \leq 2T(n/2) + cn \leq 2(cn/2) \log(n/2) + cn = cn \log_2 n - cn + cn = cn \log_2 n$.

Why is $T(n) \leq kn^2$ a "loose" bound?

Why doesn’t an attempt to prove $T(n) \leq kn$, for some $k > 0$, work?
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\]

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\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + cn \\
& \leq 2 \left( \frac{cn}{2} \log \left( \frac{n}{2} \right) \right) + cn, \text{ by the inductive hypothesis} \\
& = cn \log \left( \frac{n}{2} \right) + cn \\
& = cn \log n - cn + cn \\
& = cn \log n.
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$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(\frac{cn}{2} \log \left(\frac{n}{2}\right)\right) + cn,$$ by the inductive hypothesis

$$= cn \log \left(\frac{n}{2}\right) + cn$$

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- Why doesn’t an attempt to prove $T(n) \leq kn$, for some $k > 0$ work?
Partial Substitution

- Guess that the solution is $kn \log n$ (logarithm to the base 2).
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- Substitute guess into the recurrence relation to check what value of $k$ will satisfy the recurrence relation.
- $k \geq c$ will work.
Proof for All Values of $n$

- We assumed $n$ is a power of 2.
- How do we generalise the proof?
Proof for All Values of $n$

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- How do we generalise the proof?
- Basic axiom: $T(n) \leq T(n + 1)$, for all $n$: worst case running time increases as input size increases.
- Let $m$ be the smallest power of 2 larger than $n$.
- $T(n) \leq T(m) = O(m \log m)$
Proof for All Values of $n$

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- Basic axiom: $T(n) \leq T(n + 1)$, for all $n$: worst case running time increases as input size increases.
- Let $m$ be the smallest power of 2 larger than $n$.
- $T(n) \leq T(m) = O(m \log m) = O(n \log n)$, because $m \leq 2n$. 
Other Recurrence Relations

- Divide into \( q \) sub-problems of size \( n/2 \) and merge in \( O(n) \) time. Two distinct cases: \( q = 1 \) and \( q > 2 \).
- Divide into two sub-problems of size \( n/2 \) and merge in \( O(n^2) \) time.
Each invocation reduces the problem size by a factor of 2 ⇒ there are $\log n$ levels in the recursion tree.

At level $i$ of the tree, the problem size is $n/2^i$ and the work done is $cn/2^i$.

Therefore, the total work done is $\sum_{i=0}^{\log n} cn/2^i = O(n)$.

**Figure 5.3** Unrolling the recurrence $T(n) \leq T(n/2) + O(n)$. 
\[ T(n) = qT(n/2) + cn, \quad q = 1 \]

Each invocation reduces the problem size by a factor of 2 \( \Rightarrow \) there are \( \log n \) levels in the recursion tree.

At level \( i \) of the tree, the problem size is \( n/2^i \) and the work done is \( cn/2^i \).

Therefore, the total work done is

\[
\sum_{i=0}^{i=\log n} \frac{cn}{2^i} = O(n).
\]
\[ T(n) = qT(n/2) + cn, \quad q > 2 \]

There are \( \log_2 n \) levels in the recursion tree. At level \( i \) of the tree, there are \( q^i \) sub-problems, each of size \( n/2^i \). Therefore, the total work done is

\[
T(n) \leq \sum_{i=0}^{\log_2 n} q^i cn/2^i \leq cn \sum_{i=0}^{\log_2 n} (q^2/2)^i = O(cn \cdot (q^2/2)^{\log_2 n}) = O(cn n \log_2 q/2) = O(n \log_2 q).
\]

**Figure 5.2** Unrolling the recurrence \( T(n) \leq 3T(n/2) + O(n) \).
\[ T(n) = qT(n/2) + cn, \quad q > 2 \]

There are \( \log n \) levels in the recursion tree.
- At level \( i \) of the tree, there are \( q^i \) sub-problems, each of size \( n/2^i \).
- The total work done at level \( i \) is \( q^i cn/2^i \). Therefore, the total work done is

\[
T(n) \leq \sum_{i=0}^{i=\log_2 n} q^i \frac{cn}{2^i} \leq \]

**Figure 5.2** Unrolling the recurrence \( T(n) \leq 3T(n/2) + O(n) \).
There are \( \log n \) levels in the recursion tree.

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The total work done at level \( i \) is \( q^i cn/2^i \). Therefore, the total work done is

\[
T(n) \leq \sum_{i=0}^{i=\log_2 n} \frac{cn}{2^i} \leq cn \sum_{i=0}^{i=\log_2 n} \left( \frac{q}{2} \right)^i
\]

\[
= O\left( cn \left( \frac{q}{2} \right)^{\log_2 n} \right) = O\left( cn \left( \frac{q}{2} \right)^{(\log_{q/2} n)(\log_2 q)/2} \right)
\]

\[
= O\left( cn n^{\log_2 q/2} \right) = O\left( n^{\log_2 q} \right).
\]
\[ T(n) = 2T(n/2) + cn^2 \]

Total work done is

\[
\sum_{i=0}^{\log n} 2^i \left( \frac{cn}{2^i} \right)^2 \leq \]

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- Total work done is

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\sum_{i=0}^{\log n} 2^i \left( \frac{cn}{2^i} \right)^2 \leq O(n^2).
\]