Greedy Algorithms

T. M. Murali

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Algorithm Design

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- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

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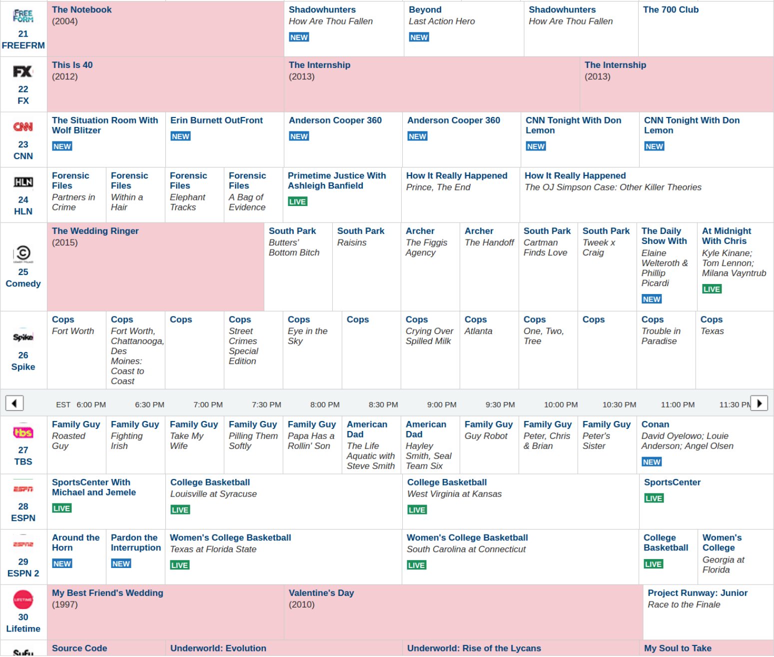
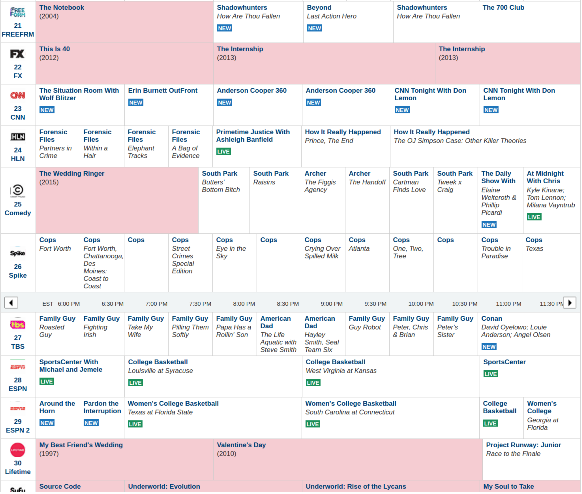
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.


FREEFORM 21 FREEFRM
The Notebook (2004)
Shadowhunters How Are Thou Fallen NEW
Beyond Last Action Hero NEW
Shadowhunters How Are Thou Fallen
The 700 Club
FX 22 FX
This Is 40 (2012)
The Internship (2013)
The Internship (2013)
CNN 23 CNN
The Situation Room With Wolf Blitzer NEW
Erin Burnett OutFront NEW
Anderson Cooper 360 NEW
Anderson Cooper 360 NEW
CNN Tonight With Don Lemon NEW
CNN Tonight With Don Lemon NEW
HLN 24 HLN
Forensic Files Partners in Crime
Forensic Files Within a Hair
Forensic Files Elephant Tracks
Forensic Files A Bag of Evidence
Primetime Justice With Ashleigh Banfield LIVE
How It Really Happened Prince, The End
How It Really Happened The OJ Simpson Case: Other Killer Theories
25 Comedy
The Wedding Ringer (2015)
South Park Butters' Bottom Bitch
South Park Raisins
Archer The Figgis Agency
Archer The Handoff
South Park Cartman Finds Love
South Park Tweek x Craig
The Daily Show With Elaine Welteroth & Phillip Picardi NEW
At Midnight With Chris Kyle Kinane; Tom Lennon; Milana Vayntrub LIVE
Spike 26 Spike
Cops Fort Worth
Cops Fort Worth, Chattanooga, Des Moines: Coast to Coast
Cops
Cops Street Crimes Special Edition
Cops Eye in the Sky
Cops
Cops Crying Over Spilled Milk
Cops Atlanta
Cops One, Two, Tree
Cops
Cops Trouble in Paradise
Cops Texas
EST 6:00 PM 6:30 PM 7:00 PM 7:30 PM 8:00 PM 8:30 PM 9:00 PM 9:30 PM 10:00 PM 10:30 PM 11:00 PM 11:30 PM
tbs 27 TBS
Family Guy Roasted Guy
Family Guy Fighting Irish
Family Guy Take My Wife
Family Guy Pilling Them Softly
Family Guy Papa Has a Rollin' Son
American Dad The Life Aquatic with Steve Smith
American Dad Hayley Smith, Seal Team Six
Family Guy Guy Robot
Family Guy Peter, Chris & Brian
Family Guy Peter's Sister
Conan David Oyelowo; Louie Anderson; Angel Olsen NEW
ESPN 28 ESPN
SportsCenter With Michael and Jemele LIVE
College Basketball Louisville at Syracuse LIVE
College Basketball West Virginia at Kansas LIVE
SportsCenter LIVE
ESPN2 29 ESPN 2
Around the Horn NEW
Pardon the Interruption NEW
Women's College Basketball Texas at Florida State LIVE
Women's College Basketball South Carolina at Connecticut LIVE
College Basketball LIVE
Women's College Georgia at Florida
30 Lifetime
My Best Friend's Wedding (1997)
Valentine's Day (2010)
Project Runway: Junior Race to the Finale
Source Code
Underworld: Evolution
Underworld: Rise of the Lycans
My Soul to Take



- Input: Start and end time of each movie.
- Constraint: Only one TV $\Rightarrow$ cannot watch two overlapping classes at the same time.
- Output: Compute the largest number of movies we can watch.

Interval Scheduling

INTERVAL SCHEDULING

INSTANCE: Nonempty set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- For any input set of jobs, algorithm must provably compute the largest set of compatible jobs.

Template for Greedy Algorithm

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
- Key question: in what order should we process the jobs?

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- Key question: in what order should we process the jobs?

  Earliest start time Increasing order of start time $s(i)$.

  Earliest finish time Increasing order of finish time $f(i)$.

  Shortest interval Increasing order of length $f(i) - s(i)$.

  Fewest conflicts Increasing order of the number of conflicting jobs. How fast can you compute the number of conflicting jobs for each job?

Greedy Ideas that Do Not Work

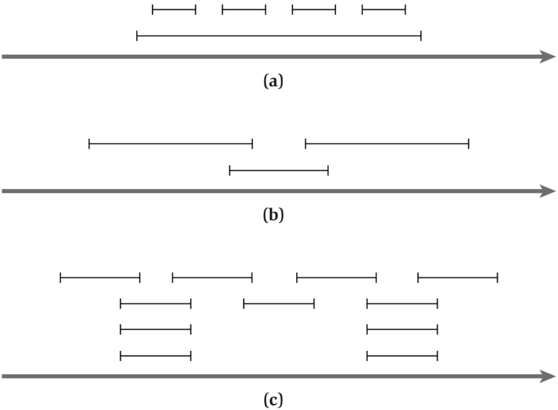

Figure 4.1 Some instances of the Interval Scheduling Problem on which natural greedy algorithms fail to find the optimal solution. In (a), it does not work to select the interval that starts earliest; in (b), it does not work to select the shortest interval; and in (c), it does not work to select the interval with the fewest conflicts.

Interval Scheduling Algorithm: Earliest Finish Time

- Schedule jobs in order of earliest finish time (EFT).

```
Initially let R be the set of all requests, and let A be empty
While R is not yet empty
  Choose a request i ∈ R that has the smallest finishing time
  Add request i to A
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Return the set A as the set of accepted requests
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- Claim: A is a compatible set of jobs. Proof follows by construction, i.e., the algorithm computes a compatible set of jobs.

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- We need to prove that $|A|$ (the number of jobs in A) is the largest possible in *any* set of mutually compatible jobs.

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  - How do we measure progress of the algorithm?

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- Proof idea 2: at each step, can we show algorithm has the "better" solution than any other answer?
  - What does "better" mean?
  - How do we measure progress of the algorithm?
- Basic idea of proof:
  - We can sort jobs in any solution in increasing order of their finishing time.
  - Finishing time of job number r selected by $A \leq$ finishing time of job number r selected by any other algorithm.

Analysing the EFT Algorithm

- Let O be an optimal set of jobs. We will show that $|A| = |O|$.
- Let $i_1, i_2, \ldots, i_k$ be the set of jobs in A in order.
- Let $j_1, j_2, \ldots, j_m$ be the set of jobs in O in order, $m \geq k$.
- Claim: For all indices $r \leq k$, $f(i_r) \leq f(j_r)$.

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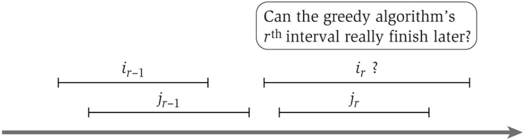

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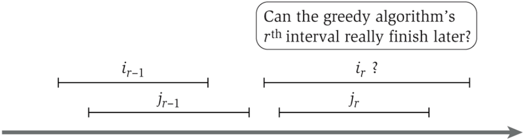

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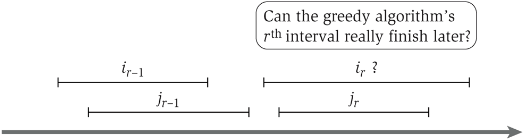

Figure 4.3 The inductive step in the proof that the greedy algorithm stays ahead.

- Claim: $m = k$.
- Claim: The greedy algorithm returns an optimal set A.

Implementing the EFT Algorithm

1. Reorder jobs so that they are in increasing order of finish time.
2. Store starting time of jobs in an array S.
3. $k = 1$.
4. While $k \leq |S|$,
   1. Output job k.
   2. Let finish time of job k be f.
   3. Iterate over S from index k onwards to find the first index i such that $S[i] \geq f$.
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- Must be careful to iterate over S such that we never scan same index more than once.
- Running time is $O(n \log n)$, dominated by sorting.

12616	CS-4104	Data and Algorithm Analysis	L	3	75	CA Shaffer	T R	2:00PM	3:15PM	SURGE 107	14T
18154	CS-4104	Data and Algorithm Analysis	L	3	70	TM Murali	M W	2:30PM	3:45PM	SURGE 104C	14M
12617	CS-4114	Formal Languages	L	3	75	L Zhang	T R	9:30AM	10:45AM	MCB 129	09T
18155	CS-4204	Computer Graphics	L	3	36	D Gracanin	T R	11:00AM	12:15PM	MCB 224	11T
19593	CS-4264	Principles Computer Security	L	3	50	KE Giles	M W	2:30PM	3:45PM	GOODW 135	14M
12618	CS-4284	Systems & Networking Capstone	L	3	40	GV Back	M W	2:30PM	3:45PM	MCB 238	14M
18156	CS-4304	Compiler Design	L	3	50	C Jung	T R	8:00AM	9:15AM	GOODW 125	08T
12620	CS-4604	Int Data Base Mgt Sys	L	3	55	RJ Quintin	M W	4:00PM	5:15PM	SURGE 109	16M
12621	CS-4624	Multimedia/Hypertext	L	3	70	EA Fox	T R	3:30PM	4:45PM	SURGE 109	15T
12622	CS-4644	Creative Computing Studio	L	3	25	SR Harrison	W	2:30PM	5:15PM	MAC 253A	14W
	Comments for CRN 12622:	Prerequisite: C or better in CS 3724 OR CS 3744									
12623	CS-4654	Intermed Data Analytics & ML	L	3	50	RB Gramacy	M W	4:00PM	5:15PM	SEITZ 313	16M
12624	CS-4704	Software Engineering Capstone	L	3	15	KR Edmison	M W	4:00PM	5:15PM	NCB 170	16M
	Comments for CRN 12624:	Prerequisite: C or better in CS 3704 OR CS 3714									
12625	CS-4784	Human-Computer Interact Capstn	L	3	30	AL Kavanaugh	F	1:00PM	3:45PM	MAC 253A	13F
	Comments for CRN 12625:	Prerequisite: CS 3724 required; CS 3714 or 3744 recommended									
19924	CS-4784	Human-Computer Interact Capstn	L	3	0	DS McCrickard	F	12:30PM	3:15PM	MCB 655	12F

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- Input: Start and end time of each class.
- Constraint: Cannot schedule two overlapping classes to the same room.
- Output: Assign each class to a room and use smallest number of rooms possible.

Interval Partitioning

INTERVAL PARTITIONING

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: A partition of the jobs into k sets, where each set of jobs is mutually compatible, and k is minimised.

- This problem models the situation where you a set of fixed jobs, and you want to schedule all jobs using as few resources as possible.

Depth of Intervals

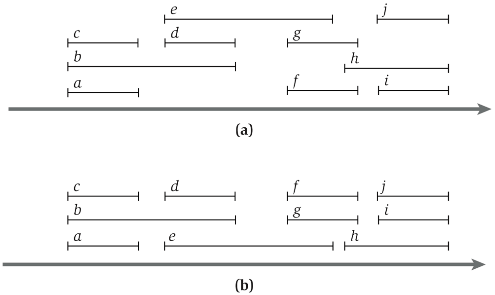
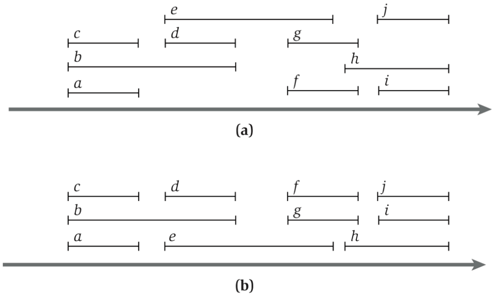


Figure 4.4 (a) An instance of the Interval Partitioning Problem with ten intervals (*a* through *j*). (b) A solution in which all intervals are scheduled using three resources: each row represents a set of intervals that can all be scheduled on a single resource.

- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.

Depth of Intervals



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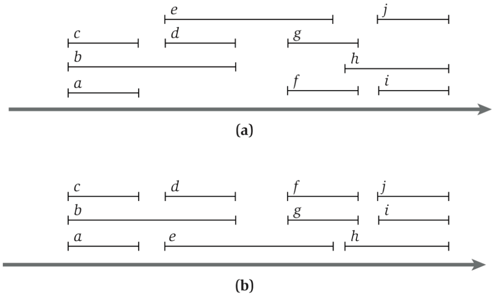

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- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.
- Claim: In any instance of INTERVAL PARTITIONING, $k \geq$ depth.
- Is it possible to compute the depth efficiently? Is $k =$ depth?

Computing the Depth of the Intervals

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1. Sort the start times and finish times of the jobs into a single list L.
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3. For i ranging from 1 to $2n$
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- Algorithm runs in $O(n \log n)$ time.

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- First, compute the depth d of the intervals.

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Sort the intervals by their start times, breaking ties arbitrarily
Let I_1, I_2, ..., I_n denote the intervals in this order
For j = 1, 2, 3, ..., n
  For each interval I_i that precedes I_j in sorted order and overlaps it
     Exclude the label of I_i from consideration for I_j
  Endfor
  If there is any label from {1, 2, ..., d} that has not been excluded then
    Assign a nonexcluded label to I_j
  Else
    Leave I_j unlabeled
  Endif
Endfor
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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.

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- The running time of the algorithm is $O(n \log n)$.

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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.
- Claim: The greedy algorithm is optimal.
- The running time of the algorithm is $O(n \log n)$. Can modify algorithm for computing depth to maintain set of available labels and to assign them efficiently.

Scheduling to Minimise Lateness

- Study different model: job i has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all n jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.

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- A job i is *delayed* if $f(i) > d(i)$; the *lateness of the job* is

$$\max(0, f(i) - d(i)).$$

- The *lateness of a schedule* is

$$\max_{1 \leq i \leq n} \Big(\max \big(0, f(i) - d(i)\big)\Big).$$

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MINIMISE LATENESS
INSTANCE: Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.
SOLUTION: Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_{1 \leq i \leq n} \big(\max \big(0, s(i) + t(i) - d(i)\big)\big)$ is as small as possible.

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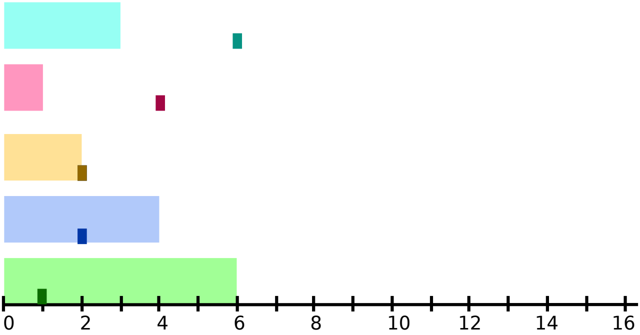
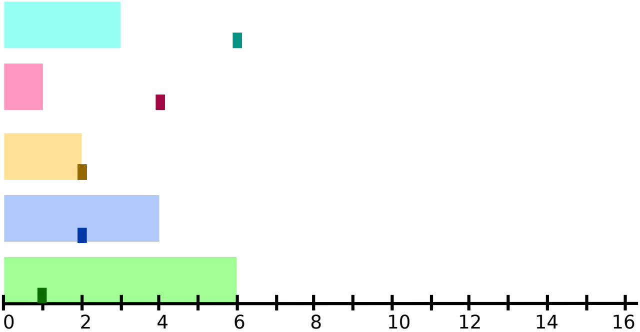


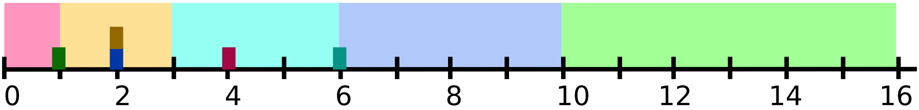
Scheduling to Minimise Lateness

MINIMISE LATENESS

INSTANCE: Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.

SOLUTION: Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_{1 \leq i \leq n} \left(\max \left(0, s(i) + t(i) - d(i)\right)\right)$ is as small as possible.





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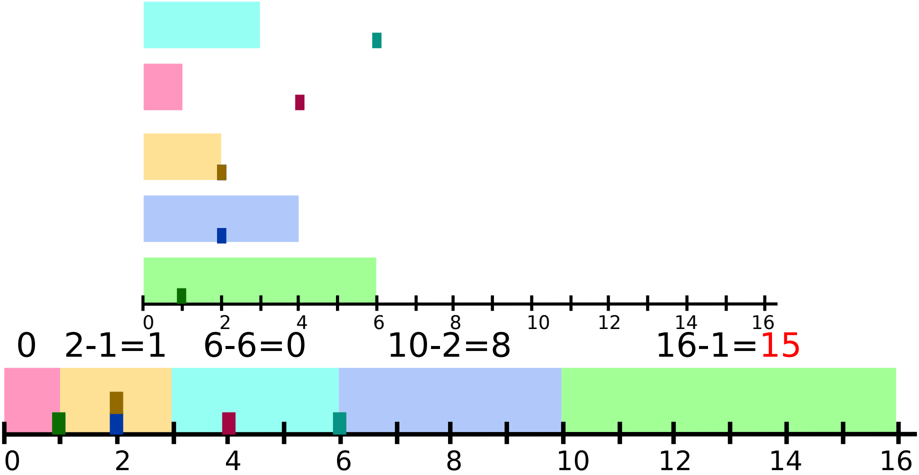

Template for Greedy Algorithm

- Key question: In what order should we schedule the jobs?

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  Earliest deadline Increasing order of deadline $d(i)$.

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$t(i)$	1	10
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  Earliest deadline Increasing order of deadline $d(i)$. Correct? Does it make sense to tackle jobs with earliest deadlines first?

Minimising Lateness: Earliest Deadline First

```
Order the jobs in order of their deadlines
Assume for simplicity of notation that d_1 ≤ ... ≤ d_n
Initially, f = s
Consider the jobs i = 1, ..., n in this order
   Assign job i to the time interval from s(i) = f to f(i) = f + t_i
   Let f = f + t_i
End
Return the set of scheduled intervals [s(i), f(i)] for i = 1, ..., n
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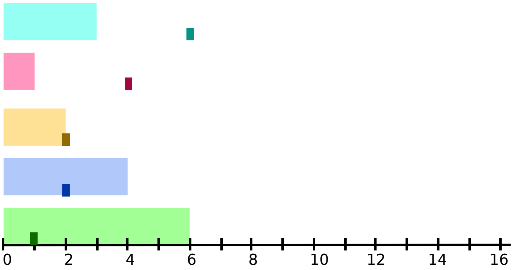

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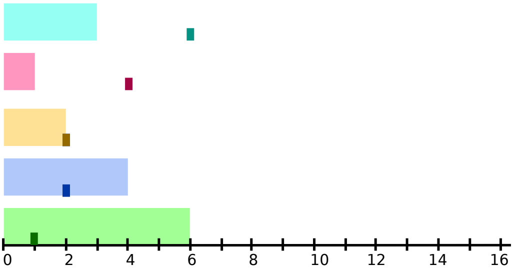

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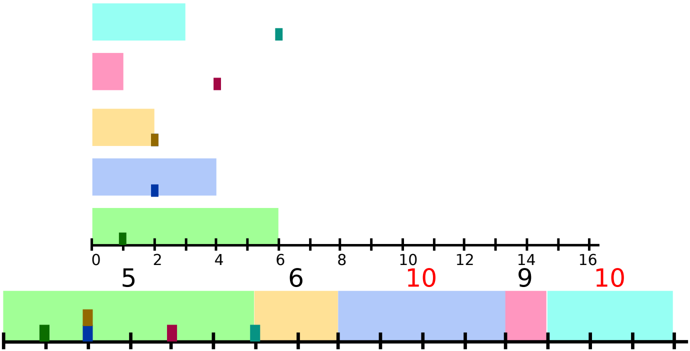
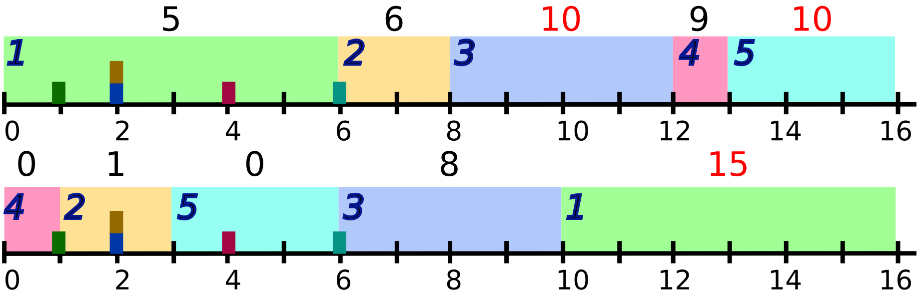


Inversions



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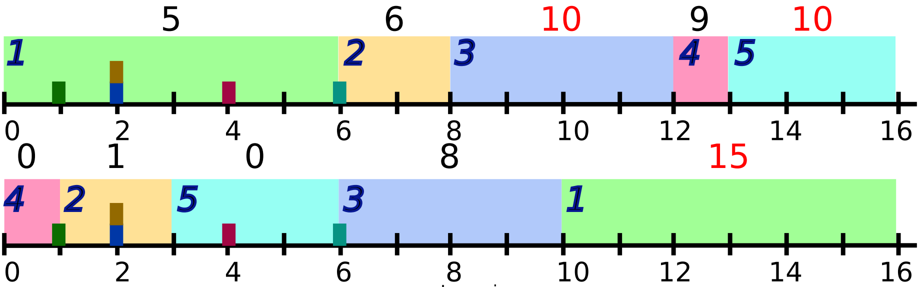

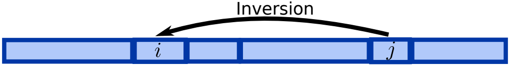

$d(j) < d(i)$ but $s(i) < s(j)$

- A schedule has an *inversion* if a job i with deadline $d(i)$ is scheduled before a job j with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
  - If i and j have the same deadlines, they cannot cause an inversion.
  - Examples:

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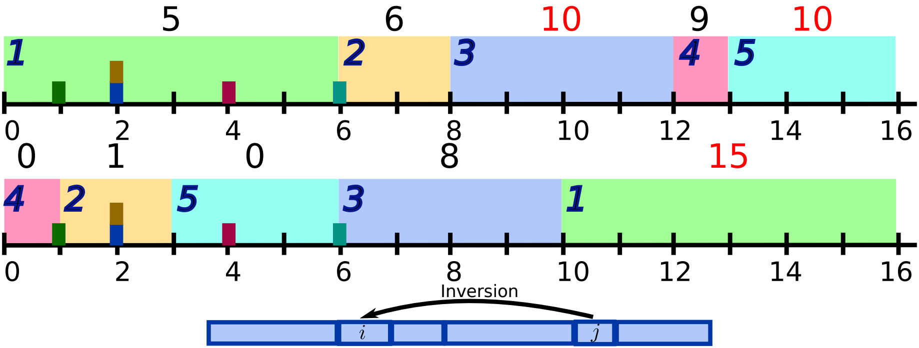

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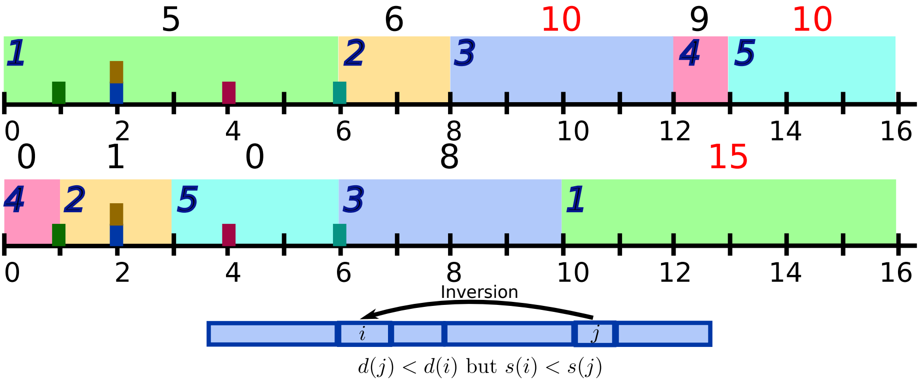
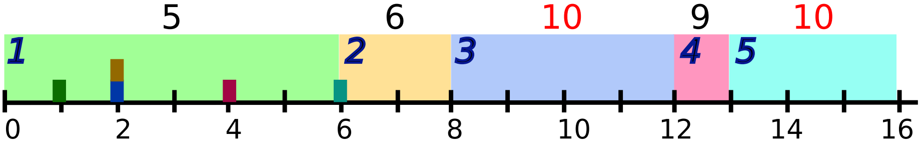


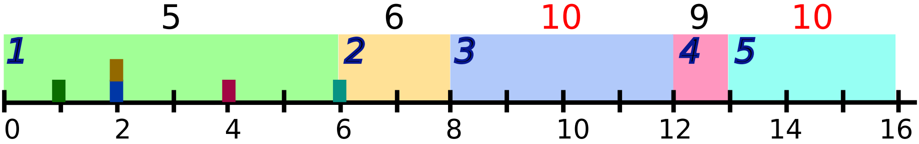
- A schedule has an *inversion* if a job i with deadline $d(i)$ is scheduled before a job j with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
  - If i and j have the same deadlines, they cannot cause an inversion.
  - Examples: 4 and 1, 2 and 1, 5 and 1, 3 and 1, 4 and 3.
- Claim: If a schedule has an inversion, then there is a pair of jobs i and j such that j is scheduled just after i and $d(j) < d(i)$.

Properties of Schedules



- Claim 1: The algorithm produces a schedule with no inversions and no idle time.

Properties of Schedules



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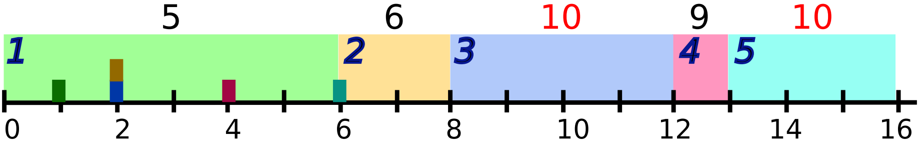

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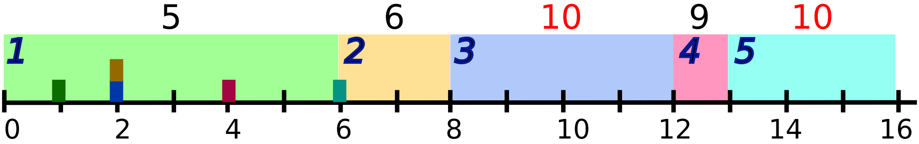

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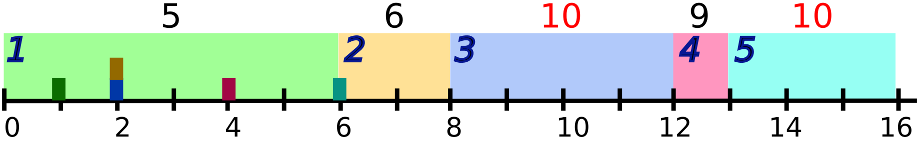

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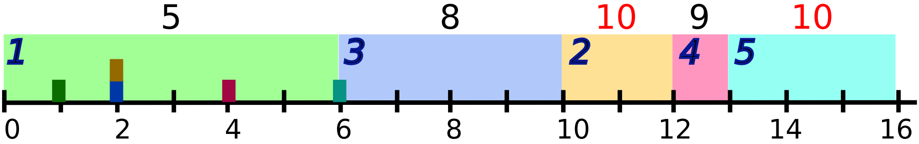

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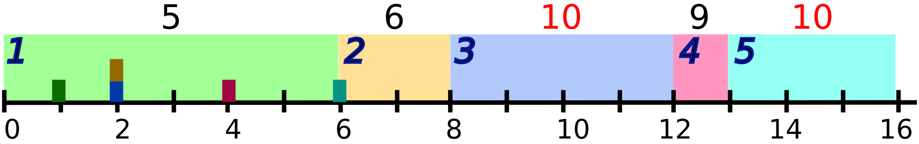
Properties of Schedules



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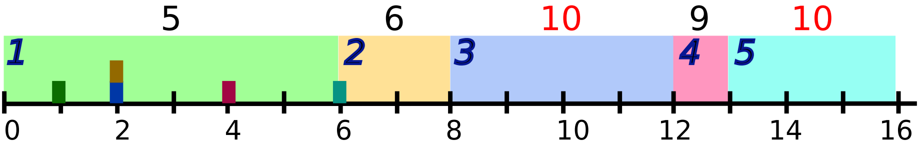

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- Claim 4: There is an optimal schedule with no inversions and no idle time.

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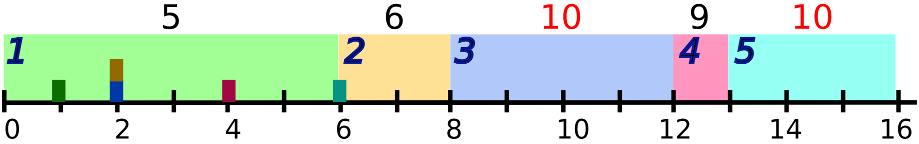

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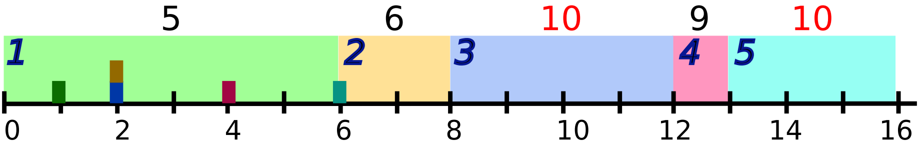

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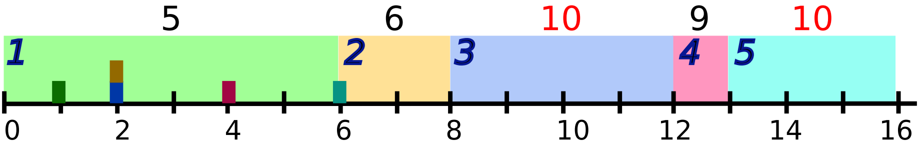

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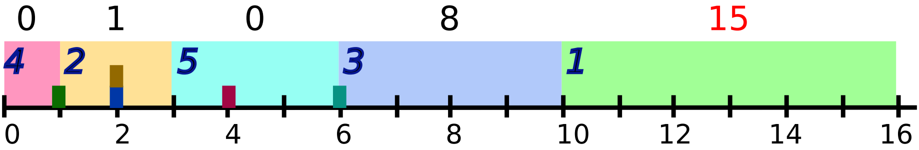
Proving Claim 4



- Claim 4: There is an optimal schedule with no inversions and no idle time.

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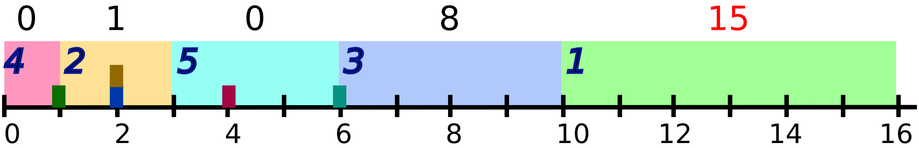

- Claim 4: There is an optimal schedule with no inversions and no idle time.
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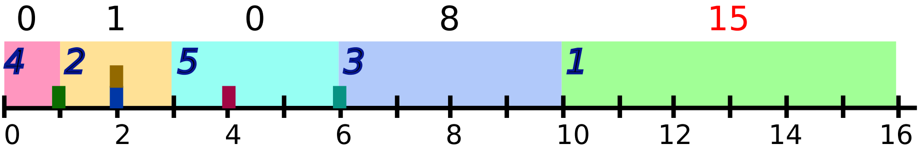

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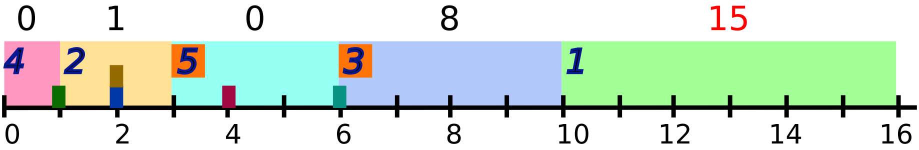

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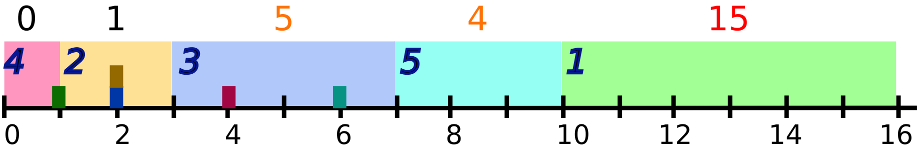

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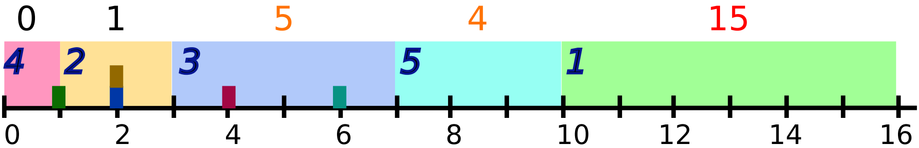

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  2. Claim: The lateness of O' is no larger than the lateness of O.
- It is enough to prove the last item, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose lateness is no larger than that of O.

Swapping Inverted Jobs

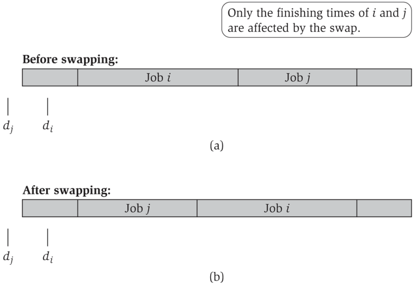

Figure 4.6 The effect of swapping two consecutive, inverted jobs.

- In O, assume each job r is scheduled for the interval $[s(r), f(r)]$ and has lateness $l(r)$. For O', let the lateness of job r be $l'(r)$.

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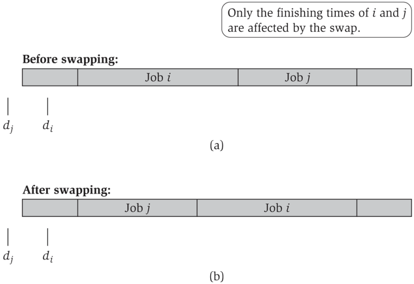

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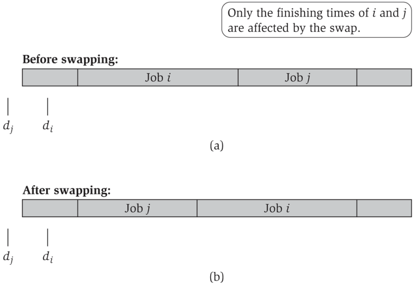

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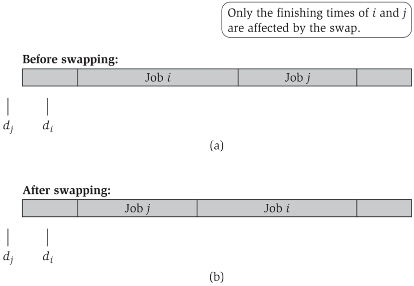

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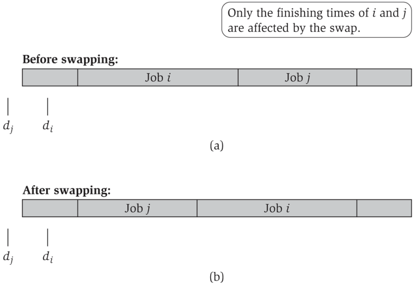

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- Claim: $l'(i) \leq l(j)$ because $l'(i) = f(j) - d_i \leq f(j) - d_j = l(j)$.

Summary of Proof

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5. Find an inversion in X and then isolate the inversion to be between consecutive jobs in X.
6. Swap the jobs to get a new schedule X_{i-1}. Where does X_{i-1} lie?

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1. Think of a schedule as a 2D point: x-coordinate is the number of inversions in the schedule and y-coordinate is the lateness of the schedule.
2. Where does the schedule A produced by the algorithm lie? Somewhere on the y-axis since it has no inversions, say $(0, l_A)$.
3. Where does some other schedule B with no inversions lie? Also at $(0, l_A)$ since all schedules with no inversions have the same lateness.
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8. Repeat one more step: X_0 has no inversions. What is X_0's location?

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9. We have a contradiction!
10. Lateness of A cannot be larger than that of O!

Common Mistakes with Exchange Arguments

- Wrong: start with algorithm's schedule A and argue that A cannot be improved by swapping two jobs.
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- Wrong: Swap two jobs that are not neighbouring in O. Pitfall is that the completion times of all intervening jobs changes.
- Correct: Show that an inversion exists between two neighbouring jobs and swap them.

Summary

- Greedy algorithms make local decisions.
- Three analysis strategies:

  Greedy algorithm stays ahead Show that after each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.

  Structural bound First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.

  Exchange argument Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.