Graphs

T. M. Murali

February 1, 3, 6, 8, 2017
The Oracle of Bacon
Basic Definitions
Graph Traversal
BFS
DFS
All Components
Implementations

On Twitter, A follows B
Selected connections highlighted below

Martha Stewart
P. Diddy
Mariah Carey
Kevin Rose
(Digg founder)

Jimmy Fallon
José Carseco

Kona
Endurance products

Emeril Lagasse

Mario Lavandeira
(Perez Hilton)

Ryan Seacrest

Solange Knowles

O-Tip

Jane Fonda

Michael Phelps

Snoop Dogg

Snoop Dogg:
"Whatz crackn nephew!!!!"

Newt Gingrich

Erykah Badu

Spicy Pants
(Celebrity gossip blogger)

Gov Bobby Jindal

Darth Vader

Digg

Intern Meredith
(TV news intern, Columbus, Ohio)

Harley Wonderpug

William Shatner

Toby Young
(1st account)

Stephen Fry

Ryan Seacrest

José Carseco

Mobile Missionary
(Christian blog)
First trophic level: Photosynthesizers

Second trophic level: Decomposers, Mutualists, Pathogens, parasites, Root-feeders

Third trophic level: Shredders, Predators, Grazers

Fourth trophic level: Higher level predators

Fifth and higher trophic levels: Higher level predators
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Definition of a Graph

- **Undirected graph** $G = (V, E)$: set $V$ of nodes and set $E$ of edges, where $E \subseteq V \times V$. Elements of $E$ are unordered pairs.
  - Say that edge $e$ is *incident* on $u$ and on $v$.
  - Exactly one edge between any pair of nodes.
  - $G$ contains no self-loops, i.e., no edges of the form $(u, u)$. 
Definition of a Graph

- **Directed graph** $G = (V, E)$: set $V$ of nodes and set $E$ of edges, where $E \subseteq V \times V$. Elements of $E$ are ordered pairs.
  - $e = (u, v)$: $u$ is the **tail** of the edge $e$, $v$ is its **head**; $e$ is **directed from** $u$ **to** $v$.
  - A pair of nodes \{u, v\} may be connected by two directed edges: $(u, v)$ and $(v, u)$.
  - $G$ contains no self loops.
A \( v_1 \)-\( v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).
A $v_1$-$v_k$ path in an undirected graph $G = (V, E)$ is a sequence $P$ of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that every consecutive pair of nodes $v_i, v_{i+1}, 1 \leq i < k$ is connected by an edge in $E$. 
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A path is \textit{simple} if all its nodes are distinct.
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  - All definitions carry over to directed graphs as well.
A \( v_1 - v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).

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An undirected graph \( G \) is **connected** if for every pair of nodes \( u, v \in V \), there is a path from \( u \) to \( v \) in \( G \).
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\textit{Distance} \( d(u, v) \) between two nodes \( u \) and \( v \) is the minimum number of edges in any \( u \)-\( v \) path.
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Trees

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Rooting a tree $T$: pick some node $r$ in the tree and orient each edge of $T$ “away” from $r$, i.e., for each node $v \neq r$, define parent of $v$ to be the node $u$ that directly precedes $v$ on the path from $r$ to $v$.

Figure 3.1 Two drawings of the same tree. On the right, the tree is rooted at node 1.
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- Node $w$ is a *child* of node $v$ if $v$ is a parent of $w$.
- Node $w$ is a *descendant* of node $v$ (or $v$ is an *ancestor* of $w$) if $v$ lies on the $r$-$w$ path.
- Node $x$ is a *leaf* if it has no descendants.
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Examples of (rooted) trees:
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Examples of (rooted) trees: organisational hierarchy, class hierarchies in object-oriented languages.

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Number of Edges in a Tree

Claim: every $n$-node tree has $n - 1$ edges.
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Proof 1:
Number of Edges in a Tree

- **Claim:** every $n$-node tree has exactly $n - 1$ edges.

- **Proof 1:** Root the tree. Each node, except the root, has a unique parent. Each edge connects one parent to one child. Therefore, the tree has $n - 1$ edges.
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- Proof 2: (by induction)
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Proof 2: (by induction) Two key pieces.
- Every tree contains at least one leaf, i.e., node of degree 1. Why?
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- Stronger claim: Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements implies the third:
  1. $G$ is connected.
  2. $G$ does not contain a cycle.
  3. $G$ contains $n - 1$ edges.
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CS4104: Graphs
Number of Edges in a Tree

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  - 1 and 2 \( \Rightarrow \) 3:
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  ▶ Note that none of these statements uses the word “tree”.
  ▶ 1 and 2 \( \Rightarrow \) 3: just proved.
  ▶ 2 and 3 \( \Rightarrow \) 1:
Number of Edges in a Tree

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- Stronger claim: Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements implies the third:
  1. $G$ is connected.
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  - Note that none of these statements uses the word “tree”.
  - 1 and 2 $\Rightarrow$ 3: just proved.
  - 2 and 3 $\Rightarrow$ 1: prove by contradiction.
  - 3 and 1 $\Rightarrow$ 2: prove yourself.
**s-t Connectivity**

**INSTANCE:** An undirected graph \( G = (V, E) \) and two nodes \( s, t \in V \).

**QUESTION:** Is there an \( s-t \) path in \( G \)?
s-t Connectivity

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**QUESTION:** Is there an $s$-$t$ path in $G$?

- The *connected component of $G$ containing $s$* is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$. 
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- Algorithm for the $s$-$t$ Connectivity problem: compute the connected component of $G$ that contains $s$ and check if $t$ is in that component.
Computing Connected Components

“Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path

Initially $R = \{s\}$

While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$

Add $v$ to $R$

Endwhile
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Endwhile
How do we implement the while loop?

- $R$ will consist of nodes to which $s$ has a path
- Initially $R = \{s\}$
- While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  - Add $v$ to $R$
- Endwhile
Issues in Computing Connected Components

How do we implement the while loop? Examine each edge in $E$. 

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Issues in Computing Connected Components

- How do we implement the while loop? Examine each edge in $E$.
- Other issues to consider:
  - Why does the algorithm terminate?
  - Does the algorithm truly compute connected component of $G$ containing $s$?
  - What is the running time of the algorithm?
Termination of the Algorithm

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]
\[ \text{Initially } R = \{s\} \]
\[ \text{While there is an edge } (u,v) \text{ where } u \in R \text{ and } v \notin R \]
\[ \quad \text{Add } v \text{ to } R \]
\[ \text{Endwhile} \]

- How many nodes does each iteration of the while loop add to \( R \)?
- How many times is the while loop executed?
Termination of the Algorithm

R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
    Add v to R
Endwhile

- How many nodes does each iteration of the while loop add to R? Exactly 1.
- How many times is the while loop executed?
Termination of the Algorithm

- How many nodes does each iteration of the while loop add to \( R \)? Exactly 1.
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- How many nodes does each iteration of the while loop add to \( R \)? Exactly 1.
- How many times is the while loop executed? At most \( n \) times.
- What is true of \( R \) at termination?
  - either \( R = V \) at the end or
  - in the last iteration, every edge either has both nodes in \( R \) or both nodes not in \( R \).
Correctness of the Algorithm

Claim: at the end of the algorithm, the set $R$ is exactly the connected component of $G$ containing $s$.  

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Claim: at the end of the algorithm, the set $R$ is exactly the connected component of $G$ containing $s$.

Proof: At termination, suppose $w \notin R$ but there is an $s$-$w$ path $P$ in $G$.

- Consider first node $v$ in $P$ not in $R$ ($v \neq s$).
- Let $u$ be the predecessor of $v$ in $P$:

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- $(u, v)$ is an edge with $u \in R$ but $v \notin R$, contradicting the stopping rule.
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- Note: wrong to assume that predecessor of $w$ in $P$ is not in $R$. 

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Recovering Paths

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- Given a node \( t \in R \), how do we recover the \( s \)-\( t \) path?
Given a node \( t \in R \), how do we recover the \( s-t \) path?

- When adding node \( v \) to \( R \), record the edge \((u, v)\).
- What type of graph is formed by these edges?

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Given a node \( t \in R \), how do we recover the \( s-t \) path?
- When adding node \( v \) to \( R \), record the edge \((u, v)\).
- What type of graph is formed by these edges? It is a tree! Why?

\( R \) will consist of nodes to which \( s \) has a path
Initially \( R = \{s\} \)
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- What type of graph is formed by these edges? It is a tree! Why?
- To recover the \( s-t \) path, trace these edges backwards from \( t \) until we reach \( s \).
Running Time of the Algorithm

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]

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- Analyse algorithm in terms of two parameters: the number of nodes $n$ and the number of edges $m$.
- Implement the while loop by examining each edge in $E$. Running time of each loop is $O(m)$. How many while loops does the algorithm execute? At most $n$. The running time is $O(mn)$. Can we improve the running time by processing edges more carefully?
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T. M. Murali
February 1, 3, 6, 8, 2017

CS4104: Graphs
Running Time of the Algorithm

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Breadth-First Search (BFS)

- Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.
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- Given layers $L_0, L_1, \ldots, L_j$, layer $L_{j+1}$ contains all nodes that
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Properties of BFS

- We have not yet described how to compute these layers.
- Claim: For each $j \geq 1$, layer $L_j$ consists of all nodes
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Claim: For each $j \geq 1$, layer $L_j$ consists of all nodes exactly at distance $j$ from $S$. Proof
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Claim: For each \( j \geq 1 \), layer \( L_j \) consists of all nodes exactly at distance \( j \) from \( S \). Proof by induction on \( j \).

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Claim: There is a path from $s$ to $t$ if and only if $t$ is a member of some layer.

Let $v$ be a node in layer $L_{j+1}$ and $u$ be the “first” node in $L_j$ such that $(u, v)$ is an edge in $G$. Consider the graph $T$ formed by all such edges, directed from $u$ to $v$. 

▶ Why is $T$ a tree?

It is connected. The number of edges in $T$ is the number of nodes in all the layers minus 1.

▶ $T$ is called the breadth-first search tree.
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Properties of BFS
- **Non-tree edge**: an edge of $G$ that does not belong to the BFS tree $T$.
- **Claim**: Let $T$ be a BFS tree, let $x$ and $y$ be nodes in $T$ belonging to layers $L_i$ and $L_j$, respectively, and let $(x, y)$ be an edge of $G$. Then $|i - j| \leq 1$. 
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Proof by contradiction: Suppose $i < j - 1$. Node $x \in L_i \Rightarrow$ all nodes adjacent to $x$ are in layers $L_1, L_2, \ldots L_{i+1}$. Hence $y$ must be in layer $L_{i+1}$ or earlier.
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Still unresolved: an efficient implementation of BFS.
Depth-First Search (DFS)

- Explore $G$ as if it were a maze: start from $s$, traverse first edge out (to node $v$), traverse first edge out of $v$, ..., reach a dead-end, backtrack, ....
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1. Mark all nodes as “Unexplored”.
2. Invoke DFS(s).

---

**DFS**($u$):

Mark $u$ as "Explored" and add $u$ to $R$

For each edge $(u, v)$ incident to $u$

- If $v$ is not marked "Explored" then
  
  Recursively invoke DFS($v$)

- Endif

Endfor
Depth-First Search (DFS)

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- **Depth-first search tree** is a tree $T$: when DFS($v$) is invoked directly during the call to DFS($v$), add edge $(u, v)$ to $T$. 
Example of DFS
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BFS vs. DFS

- Both visit the same set of nodes but in a different order.
- Both traverse all the edges in the connected component but in a different order.
- BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.
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- Non-tree edges
  - BFS within the same level or between adjacent levels.
  - DFS connect ancestors to descendants.
Properties of DFS Trees

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Observation: All nodes marked as “Explored” between the start of DFS($u$) and its end are descendants of $u$ in the DFS tree $T$. 
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- Claim: Let $x$ and $y$ be nodes in a DFS tree $T$ such that $(x,y)$ is an edge of $G$ but not of $T$. Then one of $x$ or $y$ is an ancestor of the other in $T$. 
Properties of DFS Trees

DFS(u):
Mark u as "Explored" and add u to R
For each edge (u, v) incident to u
  If v is not marked "Explored" then
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- Observation: All nodes marked as “Explored” between the start of DFS(u) and its end are descendants of u in the DFS tree T.
- Claim: Let x and y be nodes in a DFS tree T such that (x, y) is an edge of G but not of T. Then one of x or y is an ancestor of the other in T.
- Proof: Assume, without loss of generality, that DFS(u) reached x first.
  ▶ Since (x, y) is an edge in G, it is examined during DFS(x).
  ▶ Since (x, y) ∉ T, y must be marked as “Explored” during DFS(x) but before (x, y) is examined.
  ▶ Since y was not marked as “Explored” before DFS(x) was invoked, it must be marked as “Explored” between the start and the end of DFS(x).
  ▶ Therefore, y must be a descendant of x in T.
We have discussed the component containing a particular node $s$.
Each node belongs to a component.
What is the relationship between all these components?
All Connected Components

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  1. If $G$ has an $s$-$t$ path, then the connected components of $s$ and $t$ are the same.
All Connected Components

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  1. If \( G \) has an \( s-t \) path, then the connected components of \( s \) and \( t \) are the same.
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Computing All Connected Components

1. Pick an arbitrary node $s$ in $G$.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say $v$, not already visited) and repeat the BFS from $v$.
4. Repeat this process until all nodes are visited.
Representing Graphs

Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.

- Size of the graph is defined to be $m + n$.
- Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$. 
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- Assume $V = \{1, 2, \ldots, n - 1, n\}$.
- **Adjacency matrix** representation: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
  - Space used is $\Theta(n^2)$, which is optimal in the worst case.
- Check if there is an edge between node $i$ and node $j$ in $O(1)$ time.
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- Adjacency list representation: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$.
  - $n_v$ the number of neighbours of node $v$.
  - Space used is $O(n + \sum v \in G n_v) = O(n + m)$, which is optimal for every graph.
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  - Check if there is an edge between node $u$ and node $v$ in $O(n_u)$ time.
  - Iterate over all the edges incident on node $u$ in $\Theta(n_u)$ time.
Data Structures for Implementation

• “Implementation” of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.

• Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.

• How do we store the set of visited nodes? Order in which we process the nodes is crucial.
“Implementation” of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.

Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.

How do we store the set of visited nodes? Order in which we process the nodes is crucial.

- BFS: store visited nodes in a queue (first-in, first-out).
- DFS: store visited nodes in a stack (last-in, first-out)
Implementing BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.

BFS(s):

Set `Discovered[s] = true` and `Discovered[v] = false` for all other `v`.

Initialize `L[0]` to consist of the single element `s`.

Set the layer counter `i = 0`.

Set the current BFS tree `T = ∅`.

While `L[i]` is not empty:

- Initialize an empty list `L[i + 1]`.
- For each node `u ∈ L[i]`:
  - Consider each edge `(u, v)` incident to `u`.
  - If `Discovered[v] = false` then
    - Set `Discovered[v] = true`.
    - Add edge `(u, v)` to the tree `T`.
    - Add `v` to the list `L[i + 1]`.
  - Endif
- Endfor
- Increment the layer counter `i` by one.
- Endwhile
Using a Queue in BFS

- Instead of storing each layer in a different list, maintain all the layers in a single queue $L$.

BFS($s$):
- Set $\text{Discovered}[s] = \text{true}$
- Set $\text{Discovered}[v] = \text{false}$, for all other nodes $v$
- Initialize $L$ to consist of the single element $s$
- While $L$ is not empty
  - Pop the node $u$ at the head of $L$
  - Consider each edge $(u, v)$ incident on $u$
  - If $\text{Discovered}[v] = \text{false}$ then
    - Set $\text{Discovered}[v] = \text{true}$
    - Add edge $(u, v)$ to the tree $T$
    - Push $v$ to the back of $L$
  - Endif
- Endwhile

Claim: Nodes in layer $i + 1$ will appear in $L$ immediately after nodes in layer $i$.

More formally: If BFS($s$) pops $(v, l_v)$ from $L$ immediately after it pops $(u, l_u)$, then either $l_v = l_u$ or $l_v = l_u + 1$. 
Using a Queue in BFS

Instead of storing each layer in a different list, maintain all the layers in a single queue \( L \).

**BFS**(*s*):
- Set \( \text{Discovered}[s] = \text{true} \)
- Set \( \text{Discovered}[v] = \text{false} \), for all other nodes *v*
- Initialize \( L \) to consist of the single element *s*
- While \( L \) is not empty
  - Pop the node *u* at the head of \( L \)
  - Consider each edge \((u, v)\) incident on *u*
  - If \( \text{Discovered}[v] = \text{false} \) then
    - Set \( \text{Discovered}[v] = \text{true} \)
    - Add edge \((u, v)\) to the tree \( T \)
    - Push *v* to the back of \( L \)
- Endif
- Endwhile

Claim: Nodes in layer \( i+1 \) will appear in \( L \) immediately after nodes in layer \( i \).
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**BFS($s$):**

1. Set $\text{Discovered}[s] = \text{true}$
2. Set $\text{Discovered}[v] = \text{false}$, for all other nodes $v$
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     - Set $\text{Discovered}[v] = \text{true}$
     - Add edge $(u, v)$ to the tree $T$
     - Push $v$ to the back of $L$
   - Endif
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Analysis of BFS Implementation

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  - Total time for all for loops: $\sum_{u \in G} O(n_u) = O(m)$ time.
  - Maintaining layer information: $O(1)$ time per node.
  - Total time is $O(n + m)$. 
Recursive DFS

DFS(u):

Mark u as "Explored" and add u to R
For each edge (u, v) incident to u
    If v is not marked "Explored" then
        Recursively invoke DFS(v)
    Endif
Endfor

Procedure has “tail recursion”: recursive call is the last step.
Recursive DFS

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Endfor

- Procedure has “tail recursion”: recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.
Implementing DFS

- Maintain a stack $S$ to store nodes to be explored.
- Maintain an array Explored and set $\text{Explored}[v] = true$ when the algorithm pops $v$ from the stack.
- Read textbook on how to construct the DFS tree.

---

**DFS(s):**

Initialize $S$ to be a stack with one element $s$

While $S$ is not empty

- Take a node $u$ from $S$
- If $\text{Explored}[u] = false$ then
  - Set $\text{Explored}[u] = true$
  - For each edge $(u, v)$ incident to $u$
    - Add $v$ to the stack $S$
- Endfor
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    - Set $\text{Explored}[u] = true$
    - For each edge $(u, v)$ incident to $u$
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  - Endfor
- Endif
- Endwhile
Implementing DFS

- Maintain a stack $S$ to store nodes to be explored.
- Maintain an array $\text{Explored}$ and set $\text{Explored}[v] = true$ when the algorithm pops $v$ from the stack.
- Read textbook on how to construct the DFS tree.

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**DFS(s):**
**Initialization:**
- Initialize $S$ to be a stack with one element $s$

**While** $S$ is not empty
- Take a node $u$ from $S$
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**DFS(s):**

1. Initialize $S$ to be a stack with one element $s$
2. While $S$ is not empty
   1. Take a node $u$ from $S$
   2. If $\text{Explored}[u] = false$ then
      1. Set $\text{Explored}[u] = true$
      2. For each edge $(u, v)$ incident to $u$
         1. Add $v$ to the stack $S$
   3. Endfor
3. Endif
4. Endwhile
Implementing DFS

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Comparing Recursion and Iteration

DFS(u):
    Mark $u$ as "Explored" and add $u$ to $R$
    For each edge $(u, v)$ incident to $u$
        If $v$ is not marked "Explored" then
            Recursively invoke DFS($v$)
        Endif
    Endfor

DFS(s):
    Initialize $S$ to be a stack with one element $s$
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Analysing DFS

DFS(s):

Initialize $S$ to be a stack with one element $s$

While $S$ is not empty

Take a node $u$ from $S$

If $\text{Explored}[u] = \text{false}$ then

Set $\text{Explored}[u] = \text{true}$

For each edge $(u, v)$ incident to $u$

Add $v$ to the stack $S$

Endfor

Endif

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How many times is a node’s adjacency list scanned?
Analysing DFS

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Analysing DFS

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Analysing DFS

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