Analysis of Algorithms

T. M. Murali

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Force-add: Visit https://www.cs.vt.edu/S17Force-Adds before 3:45pm today and use the password “4104tmm$“
What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
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**Goal**

Develop algorithms that provably run quickly and use low amounts of space.
Worst-case Running Time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$. 

Why worst-case? Why not average-case or on random inputs?

Input size = number of elements in the input.

Values in the input do not matter, except for specific algorithms.

Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.
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Polynomial Time

- Brute force algorithm: Check every possible solution.

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Brute force algorithm for sorting:

1. Try all possible $n!$ permutations of the numbers.
2. For each permutation, check if it is sorted.
3. The running time is $n n!$. Unacceptable in practice!

Desirable scaling property: When the input size doubles, the algorithm should only slow down by some constant factor $c$.

An algorithm has a polynomial running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size $n$, the running time of the algorithm is bounded by $c n^d$ steps.

Definition: An algorithm is efficient if it has a polynomial running time.
Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting?

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An algorithm has a *polynomial* running time if there exist constants \( c > 0 \) and \( d > 0 \) such that on every input of size \( n \), the running time of the algorithm is bounded by \( cn^d \) steps.
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An algorithm is *efficient* if it has a polynomial running time.
Comparing Mathematical Functions

- Assume all (mathematical) functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting:
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\[ 100n \log_2 n \leq n^2 \]
\[ 1000n \leq n^2 \]
\[ 5n^2 - 4n \geq 1000n \log n \]
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- Bubble sort and insertion sort take roughly $n^2$ comparisons while quick sort (only on average) and merge sort take roughly $n \log_2 n$ comparisons.
  - “Roughly” hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$. 

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  - “Roughly” hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$.
- How can make statements such as the following, in order to compare the running times of different algorithms?
  - $100n \log_2 n \leq n^2$
  - $10000n \leq n^2$
  - $5n^2 - 4n \geq 1000n \log n$
"10000n \leq n^2"
“\(10000n \leq n^2\)"

10000n vs. \(O(n^2)\)

Graph showing the comparison between 10000n and \(n^2\) for different values of n.
**Upper Bound**

**Definition**

**Asymptotic upper bound:** A function $f(n)$ is $O(g(n))$ if for all $n$, $f(n) \leq c \cdot g(n)$.

$10000n$ is $O(n^2)$,
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10000$n$ is $O(n^2)$, $c = 1$, $n_0 = 10000$
$100n \log_2 n$ and $n^2$

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$100n \log_2 n$ and $n^2$

$100n \log_2 n$ is $O(n^2)$, $c = 100$, $n_0 = 1$
Lower Bound

Definition

Asymptotic lower bound: A function \( f(n) \) is \( \Omega(g(n)) \) if for all \( n \), we have \( f(n) \geq c \cdot g(n) \).
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$n \log_2 n/10$ and $\Omega(n)$

- $n \log n/10$
- $n$
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\]
Meaning of “Lower Bound” in Different Contexts

- Mathematical functions:

\[ n \log n / 10 = \Omega(n) \]

This statement is purely about these two functions without relevance to any algorithm or problem.

Algorithms: The lower bound on the running time of bubble sort is \( \Omega(n^2) \).

There is some input of \( n \) numbers that will cause bubble sort to take at least \( \Omega(n^2) \) time, e.g., input the numbers in decreasing order.

Problems: The problem of sorting \( n \) numbers has a lower bound of \( \Omega(n \log n) \).

For any comparison-based sorting algorithm, there is at least one input for which that algorithm will take \( \Omega(n \log n) \) steps.
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Tight Bound

Definition

Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. 

In all these definitions, $c$ and $n_0$ are constants independent of $n$. 

Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$. 

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Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
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- If \( f = O(h) \) and \( g = O(h) \), then \( f + g = O(h) \).
- Similar statements hold for lower and tight bounds.
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- If \( f = O(g) \), then \( f + g = \Theta(g) \).
Examples

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- \( f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d) \), if \( d > 0 \) is an integer constant and \( a_d > 0 \).
  - \( O(n^d) \) is the definition of polynomial time.
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  - $O(n^d)$ is the definition of *polynomial time*.
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
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  - \( O(n^d) \) is the definition of polynomial time.
- Is an algorithm with running time \( O(n^{1.59}) \) a polynomial-time algorithm?
- \( O(\log_a n) = O(\log_b n) \) for any pair of constants \( a, b > 1 \).
- For every constant \( x > 0 \), \( \log n = O(n^x) \).
Examples

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- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every constant $x > 0$, $\log n = O(n^x)$.
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$. 
Different functions of $n$
More functions of $n$

$n$, $\log_2 n$, $\log_3 n$, $n^{0.5}$
Linear Time

- Running time is at most a constant factor times the size of the input.
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- Computing the median (or $k$th smallest) element in an unsorted list.
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Finding the minimum, merging two sorted lists.
Computing the median (or kth smallest) element in an unsorted list. “Median-of-median” algorithm.
Sub-linear time.
Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Computing the median (or $k$th smallest) element in an *unsorted* list. “Median-of-median” algorithm.
- Sub-linear time. Binary search in a sorted array of $n$ numbers takes $O(\log n)$ time.
$O(n \log n)$ Time

- Any algorithm where the costliest step is sorting.
Enumerate all pairs of elements.

- Given a set of $n$ points in the plane, find the pair that are the closest.

  Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.
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Algorithm: For each subset $S$ of $k$ nodes, check if $S$ is an independent set. If the answer is yes, report it.
Does a graph have an independent set of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that no two are joined by an edge?

Algorithm: For each subset $S$ of $k$ nodes, check if $S$ is an independent set. If the answer is yes, report it.

Running time is $O(k^2 \binom{n}{k}) = O(n^k)$. 

$O(n^k)$ Time
What is the largest size of an independent set in a graph with $n$ nodes?
Beyond Polynomial Time

What is the largest size of an independent set in a graph with $n$ nodes?

Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size $i$. Output largest independent set found.
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What is the running time?

\[ O(n^2) \]
Beyond Polynomial Time

- **What is the largest size of an independent set in a graph with** \( n \) **nodes?**
- **Algorithm:** For each \( 1 \leq i \leq n \), check if the graph has an independent size of size \( i \). Output largest independent set found.
- **What is the running time?** \( O(n^22^n) \).