Applications of Network Flow

T. M. Murali

April 7, 12 2016

Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.

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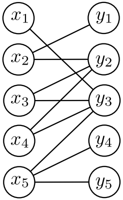
  - Multi-camera scene reconstruction.
  - Gene function prediction.
- We will only sketch proofs. Read details from the textbook.

Matching in Bipartite Graphs



- *Bipartite Graph*: a graph $G(V, E)$ where
  $V = X \cup Y$, X and Y are disjoint and
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- Bipartite graphs model situations in which objects are matched with or assigned to other objects: e.g., marriages, residents/hospitals, jobs/machines.

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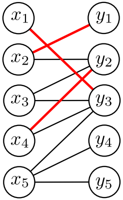

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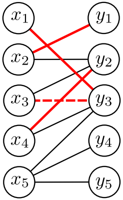

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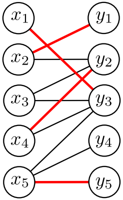

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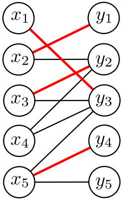

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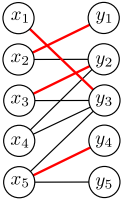

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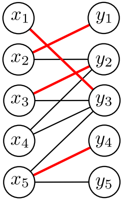

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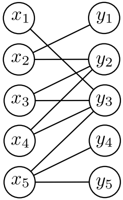
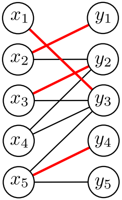
  - The graph in the figure does not have a perfect matching because both y_4 and y_5 are adjacent only to x_5.

Bipartite Graph Matching Problem



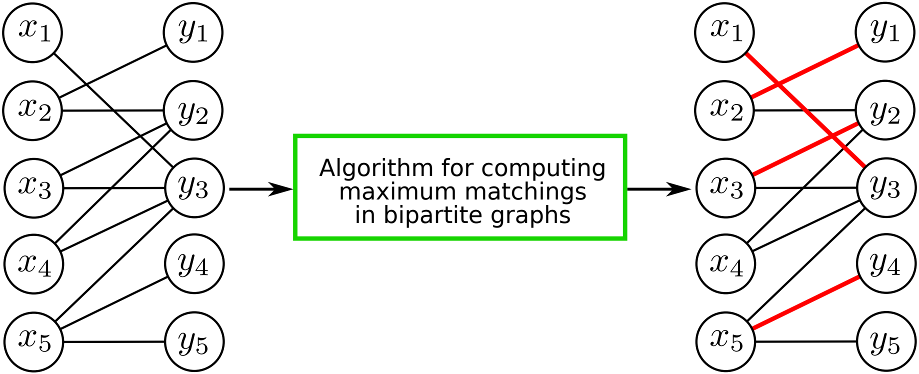


BIPARTITE MATCHING

INSTANCE: A Bipartite graph G.

SOLUTION: The matching of largest size in G.

Normal Approach for Solving a Problem



(i) Develop algorithm for computing maximum matchings in bipartite graphs.

(ii) Prove that the algorithm is correct, i.e., for every possible input, it compute the size of the largest matching in the bipartite graph accurately.

(iii) Analyze running time of the algorithm.

Alternative Approach for Solving a Problem

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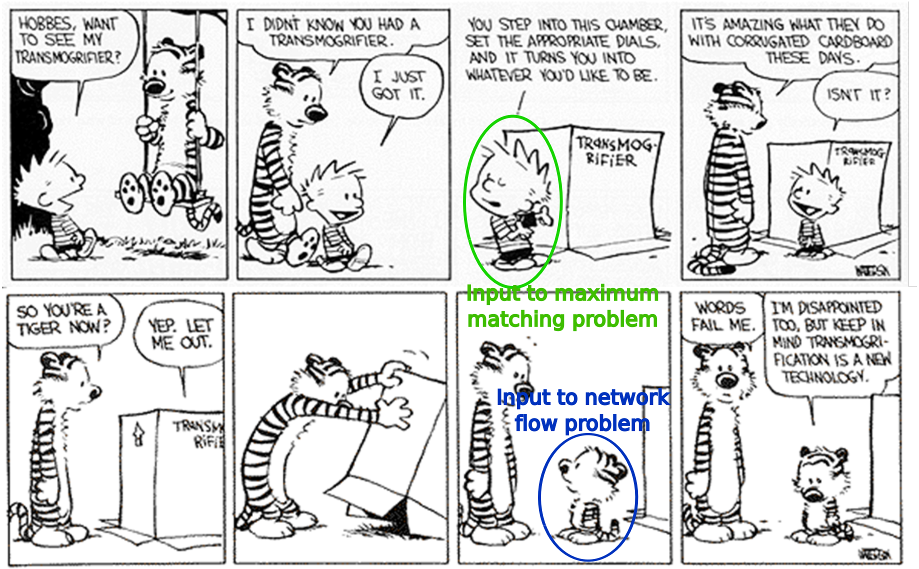

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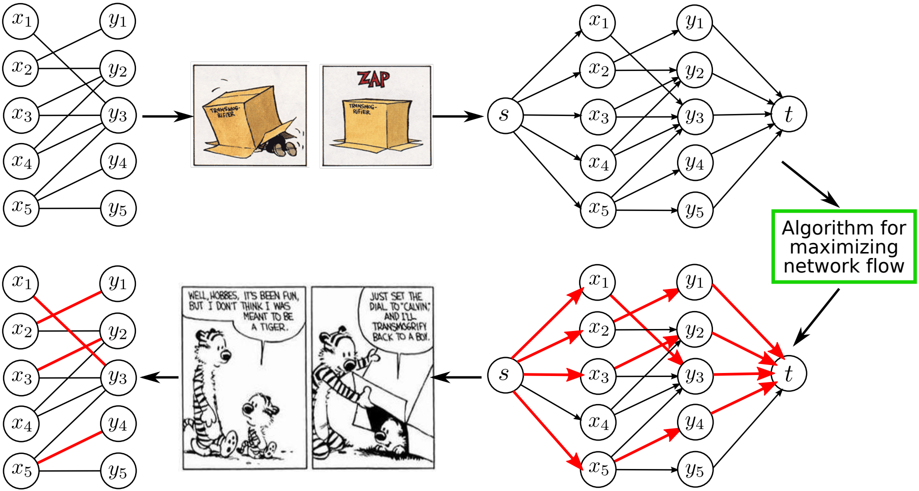
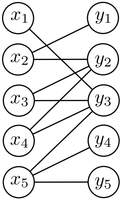
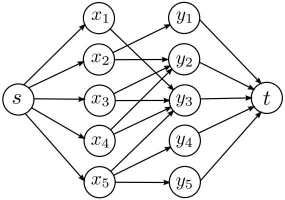


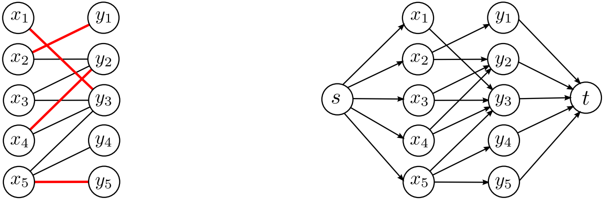
Algorithm for Bipartite Graph Matching





- Convert G to a flow network G': direct edges from X to Y, add nodes s and t, connect s to each node in X, connect each node in Y to t, set all edge capacities to 1.
- Compute the maximum flow in G'.
- Claim: the value of the maximum flow in G' is the size of the maximum matching in G.
- In general, there is matching with size k in G if and only if there is a (integer-valued) flow of value k in G'.

Correctness of Bipartite Graph Matching Algorithm



- Matching $\Rightarrow$ flow: if there is a matching with k edges in G, there is an s-t flow of value k in G'.

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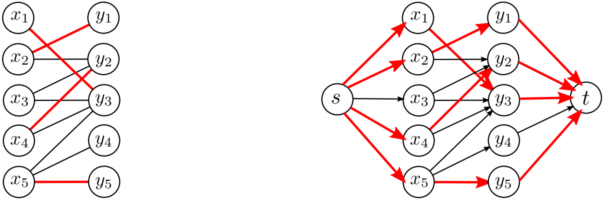

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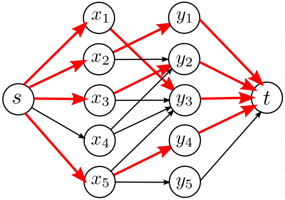

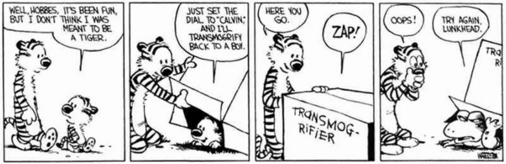

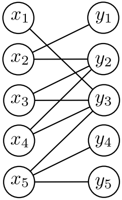

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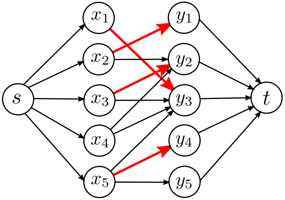

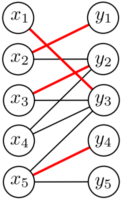

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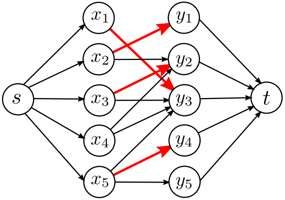

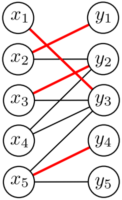

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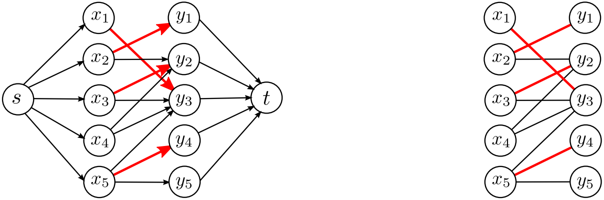

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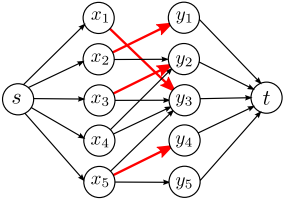

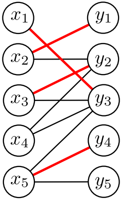

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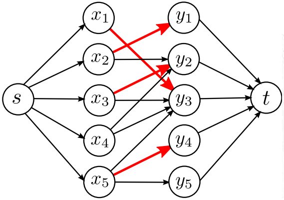

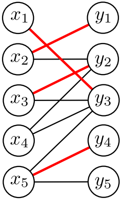

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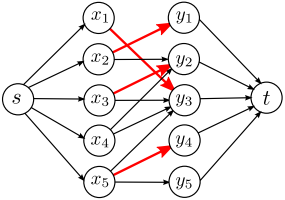

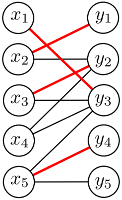

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- Read the book on what augmenting paths mean in this context.

Running time of Bipartite Graph Matching Algorithm

- Suppose G has m edges and n nodes in X and in Y.

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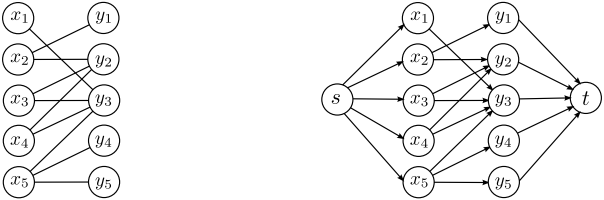
Bipartite Graphs without Perfect Matchings



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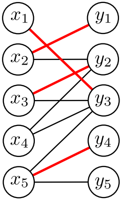

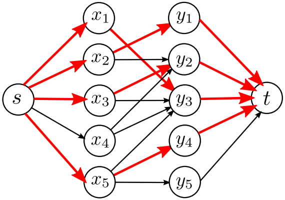

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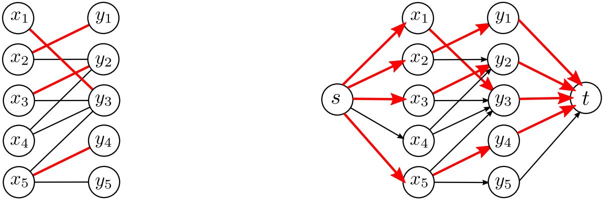

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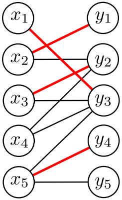

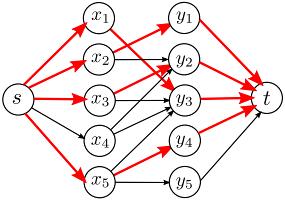

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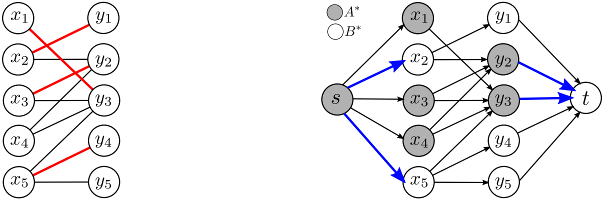

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- Suppose G has no perfect matching. Can we exhibit a short "certificate" of that fact? What can such certificates look like?
- G has no perfect matching iff there is a cut in G' with capacity less than n. Therefore, the cut is a certificate.

Bipartite Graphs without Perfect Matchings

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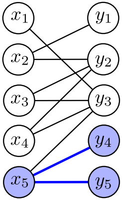

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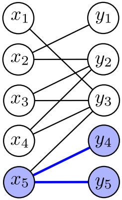

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- Hall's Theorem: Let $G(X \cup Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then G either has a perfect matching or there is a subset $A \subseteq Y$ such that $|A| > |\Gamma(A)|$. A perfect matching or such a subset can be computed in $O(mn)$ time.

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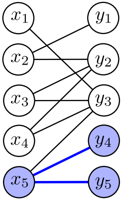

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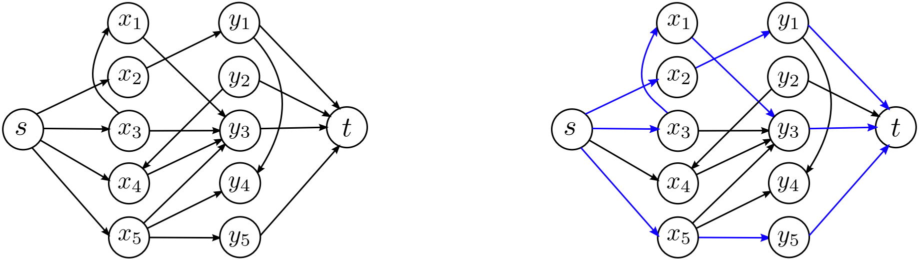
Edge-Disjoint Paths



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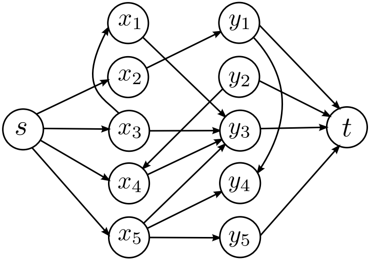

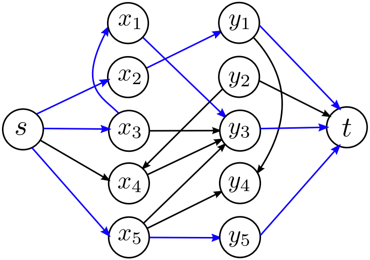

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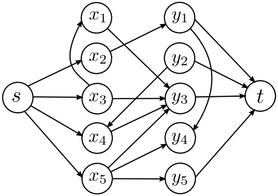
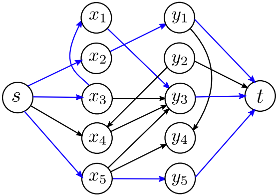
DIRECTED EDGE-DISJOINT PATHS

INSTANCE: Directed graph $G(V, E)$ with two distinguished nodes s and t.

SOLUTION: The maximum number of edge-disjoint paths between s and t.

Mapping to the Max-Flow Problem





- Convert G into a flow network: s is the source, t is the sink, each edge has capacity 1.
- Claim: There are k edge-disjoint paths from s to t in a directed graph G if and only if the maximum value of an s-t flow in G is $\geq k$.

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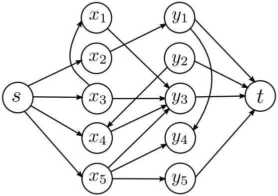

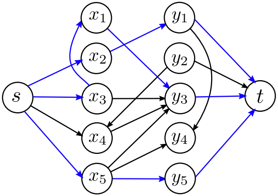

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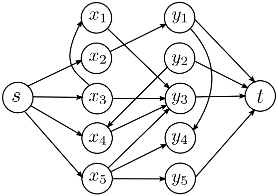

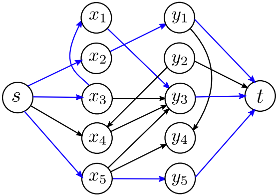

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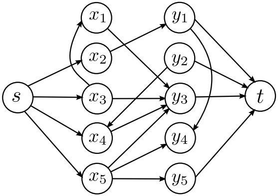

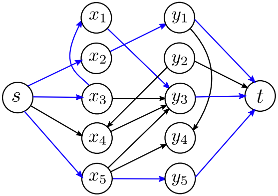

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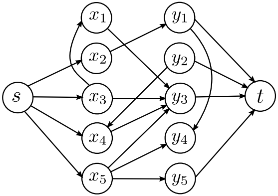

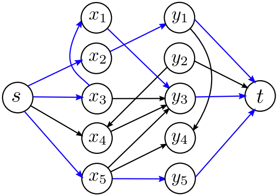

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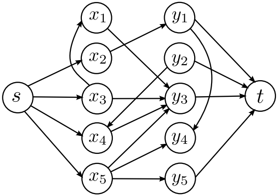

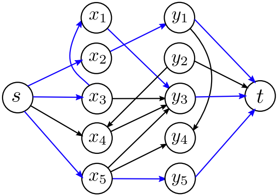

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  Inductive step: Construct a set of ν s-t paths from f. Work out on the board.
- Note: Formulating the inductive hypothesis precisely can be tricky.
- Strategy is to try to prove the inductive step first.
- During this proof, you will observe two types of "smaller" flows:
  (i) When you succeed in finding an s-t path, you get a new flow f' that is smaller, i.e., $\nu(f') < \nu$ carrying flow on fewer edges, i.e., $\kappa(f') < \kappa(f)$.
  (ii) When you run into a cycle, you get a new flow f' with $\nu(f') = \nu$ but carrying flow on fewer edges, i.e., $\kappa(f') < \kappa(f)$ edges.

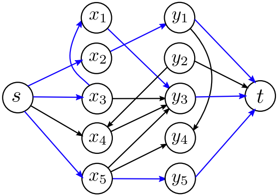
Running Time of the Edge-Disjoint Paths Algorithm

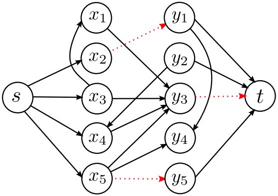
- Given a flow of value k, how quickly can we determine the k edge-disjoint paths?

Running Time of the Edge-Disjoint Paths Algorithm

- Given a flow of value k, how quickly can we determine the k edge-disjoint paths? $O(mn)$ time.
- Corollary: The Ford-Fulkerson algorithm can be used to find a maximum set of edge-disjoint s-t paths in a directed graph G in $O(mn)$ time.

Certificate for Edge-Disjoint Paths Algorithm





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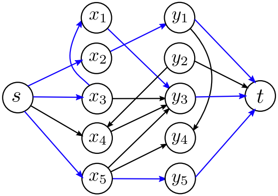
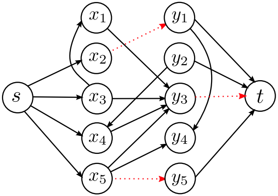
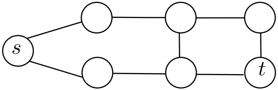




- A set $F \subseteq E$ of edge separates s and t if the graph $(V, E - F)$ contains no s-t paths.
- Menger's Theorem: In every directed graph with nodes s and t, the maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects s from t.

Edge-Disjoint Paths in Undirected Graphs



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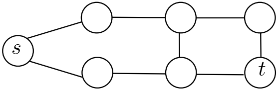

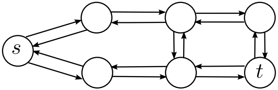

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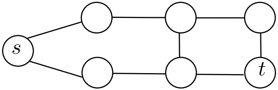

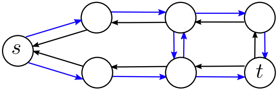

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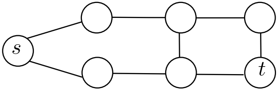

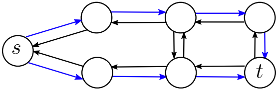

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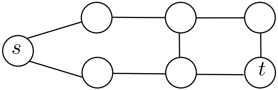

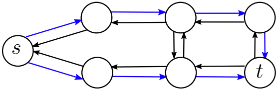

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- Can obtain an integral flow where only one of the directed counterparts of (u, v) has non-zero flow.
- We can find the maximum number of edge-disjoint paths in $O(mn)$ time.
- We can prove a version of Menger's theorem for undirected graphs: in every undirected graph with nodes s and t, the maximum number of edge-disjoint s–t paths is equal to the minimum number of edges whose removal separates s from t.

Image Segmentation


- A fundamental problem in computer vision is that of segmenting an image into coherent regions.
- A basic segmentation problem is that of partitioning an image into a foreground and a background: label each pixel in the image as belonging to the foreground or the background.
  - Note that the image on the right shows segmentation into multiple regions but we are interested in the segmentation into two regions.

Formulating the Image Segmentation Problem

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- Let E be the set of pairs of neighbouring pixels.
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- These likelihoods are specified in the input to the problem.
- We want the foreground/background boundary to be smooth: For each pair (i, j) of pixels, there is a separation penalty $p_{ij} \geq 0$ for placing one of them in the foreground and the other in the background.

The Image Segmentation Problem

IMAGE SEGMENTATION

INSTANCE: Pixel graphs $G(V, E)$, likelihood functions $a, b : V \to \mathbb{R}^+$, penalty function $p : E \to \mathbb{R}^+$

SOLUTION: *Optimum labelling*: partition of the pixels into two sets A and B that maximises

$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}.$$

Developing an Algorithm for Image Segmentation

- There is a similarity between cuts and labellings.
- But there are differences:
  - We are maximising an objective function rather than minimising it.
  - There is no source or sink in the segmentation problem.
  - We have values on the nodes.
  - The graph is undirected.

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Maximization to Minimization

- Let $Q = \sum_i (a_i + b_i)$.
- Notice that $\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j$.
- Therefore, maximising

$$\begin{aligned} q(A,B) &= \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cup \{i,j\}| = 1}} p_{ij} \\ &= Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij} \end{aligned}$$

is identical to minimising

$$q'(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

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- Solve the issues like we did earlier.
- Add a new "super-source" s to represent the foreground.
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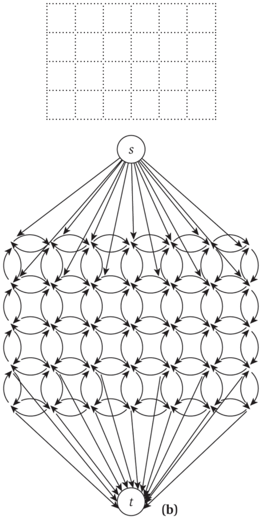
- Connect s and t to every pixel and assign capacity a_i to edge (s, i) and capacity b_i to edge (i, t).
- Direct edges away from s and into t.
- Replace each edge (i, j) in E with two directed edges of capacity p_{ij}.



(b)

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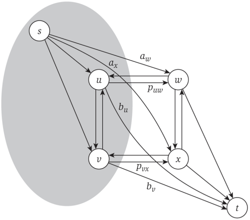

Figure 7.19 An s-t cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.

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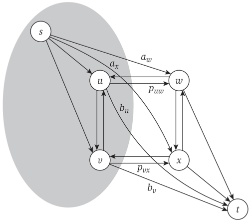

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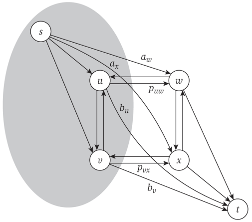

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$$c(A, B) \quad = \quad \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij} = q'(A, B).$$

Solving the Image Segmentation Problem

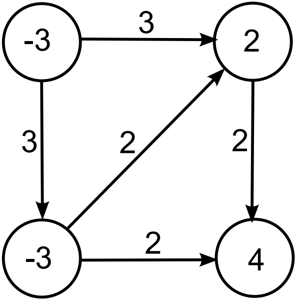
- The capacity of a s-t cut $c(A, B)$ exactly measures the quantity $q'(A, B)$.
- To maximise $q(A, B)$, we simply compute the s-t cut (A, B) of minimum capacity.
- Deleting s and t from the cut yields the desired segmentation of the image.

Extension of Max-Flow Problem

- Suppose we have a set S of multiple sources and a set T of multiple sinks.
- Each source can send flow to any sink.
- Let us not maximise flow here but formulate the problem in terms of demands and supplies.

Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:



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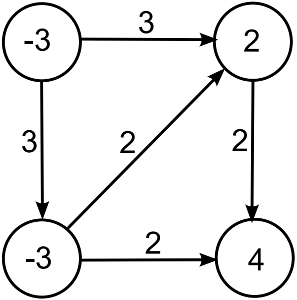

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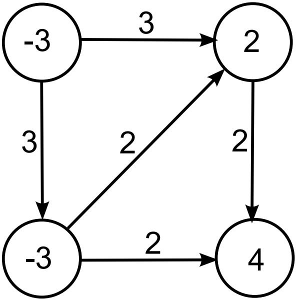
  - S is the set of nodes with negative demand and T is the set of nodes with positive demand.



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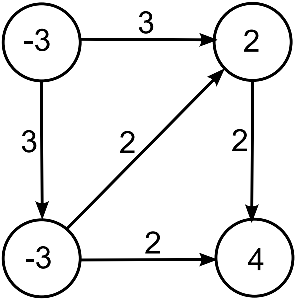

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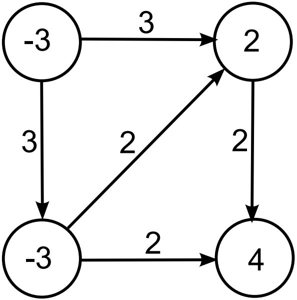

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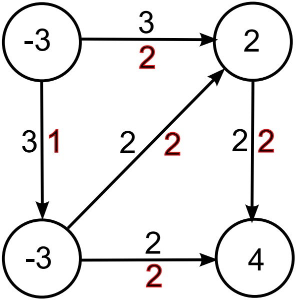

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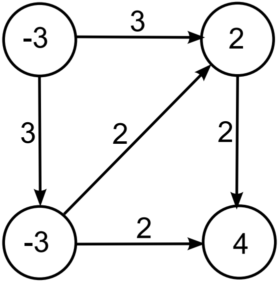
Circulation with Demands

INSTANCE: A directed graph $G(V, E)$, $c : E \rightarrow \mathbb{Z}^+$, and $d : V \rightarrow \mathbb{Z}$.

SOLUTION: Does a *feasible* circulation exist, i.e., it meets the capacity and demand conditions?

Properties of Feasible Circulations



- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$.

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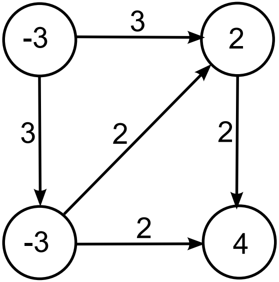


- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$.
- Corollary: $\sum_{v, d_v > 0} d_v = \sum_{v, d_v < 0} -d_v$. Let D denote this common value.

Mapping Circulation to Maximum Flow

- Create a new graph $G' = G$ and
  (i) create two new nodes in G': a source s^* and a sink t^*;
  (ii) connect s^* to each node v in S using an edge with capacity $-d_v$;
  (iii) connect each node v in T to t^* using an edge with capacity d_v.

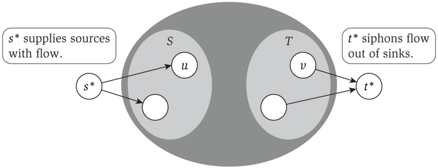
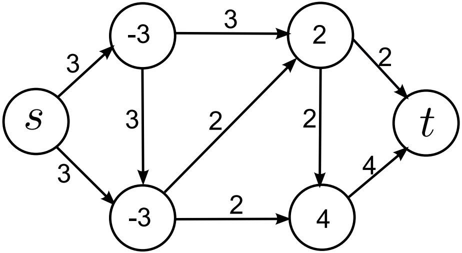
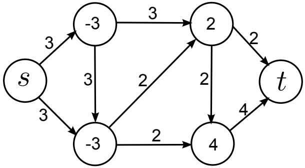
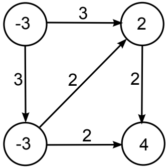


Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.



Computing a Feasible Circulation





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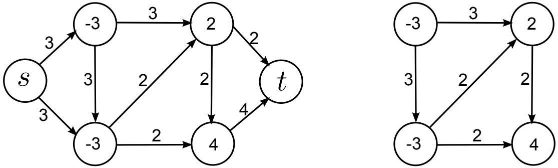

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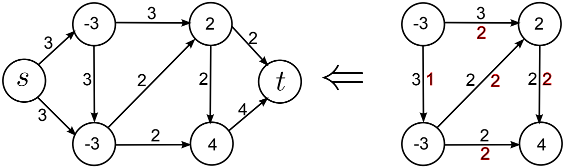

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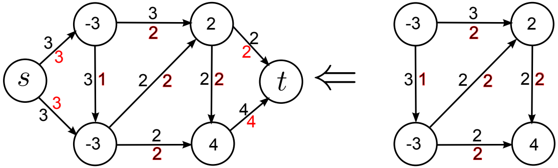

- We will look for a maximum s^*-t^* flow f in G'; $\nu(f) \leq D$.
- Circulation $\Rightarrow$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge (s^*, v) and d_v units of flow along each edge (v, t^*). The value of this flow is D. (Prove it yourself.)

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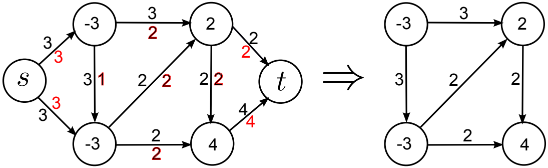

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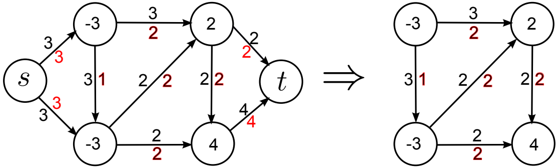

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- Flow $\Rightarrow$ circulation. If there is an s^*-t^* flow of value D in G', edges incident on s^* and on t^* must be saturated with flow. Deleting these edges from G' yields a feasible circulation in G. (Prove it yourself.)

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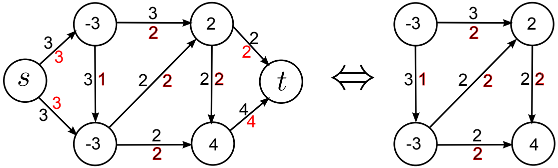

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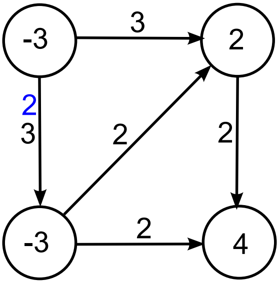
- We have proved that there is a feasible circulation with demands in G iff the maximum s^*-t^* flow in G' has value D.

Circulation with Demands and Lower Bounds



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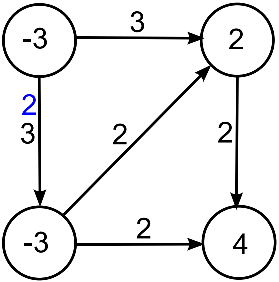

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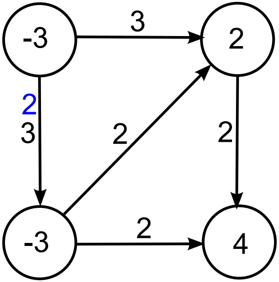

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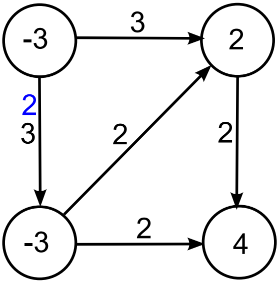

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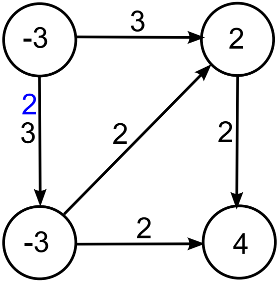

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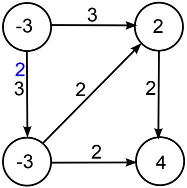
- Is there a feasible circulation?

Algorithm for Circulation with Lower Bounds



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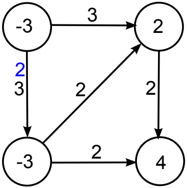

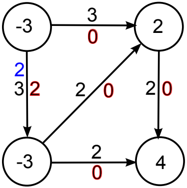

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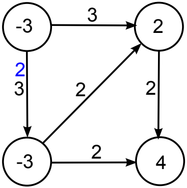

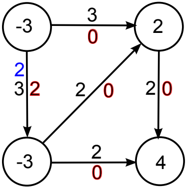

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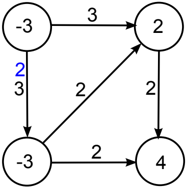

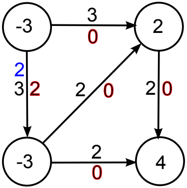

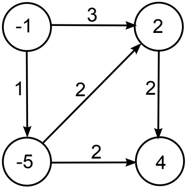

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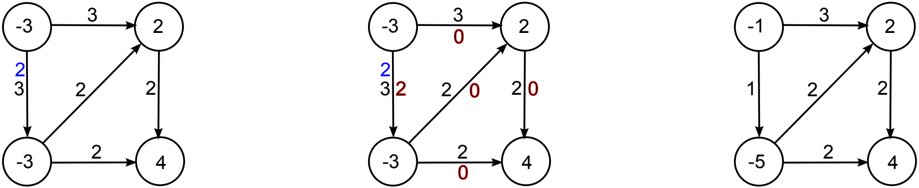

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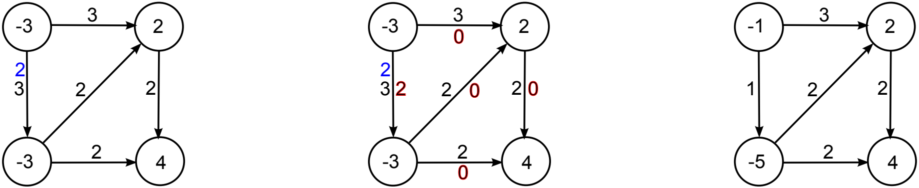
- How much capacity do we have left on each edge?

Algorithm for Circulation with Lower Bounds



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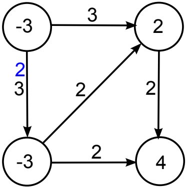

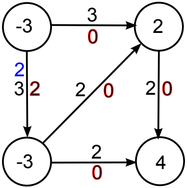

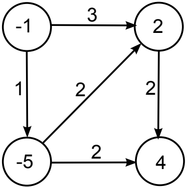

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- How much capacity do we have left on each edge? $c(e) - l(e)$.
- Approach: define a new graph G' with the same nodes and edges: each edge e has lower bound 0, capacity $c(e) - l(e)$; demand of each node v is $d_v - L_v$.
- Claim: there is a feasible circulation in G iff there is a feasible circulation in G'. Read the proof in the textbook.

Airline Scheduling

- Airlines face very complex computational problems.
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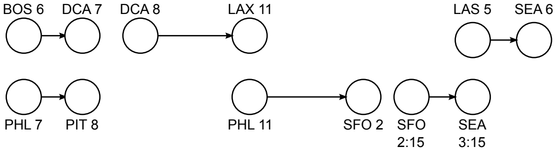
- Produce schedules for thousands of routes.
- Make these schedules efficient in terms of crew allocation, equipment usage, fuel costs, customer satisfaction, etc.
- Modelling these problems realistically is out of the scope of the course.
- We will focus on a "toy" problem that cleanly captures some of the resource allocation problems they have to deal with.

Creating Flight Schedules



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- Each flight segment (or flight) specified by four parameters: origin airport, destination airport, departure time, arrival time.

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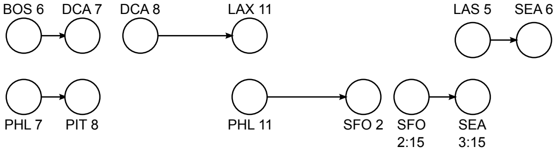
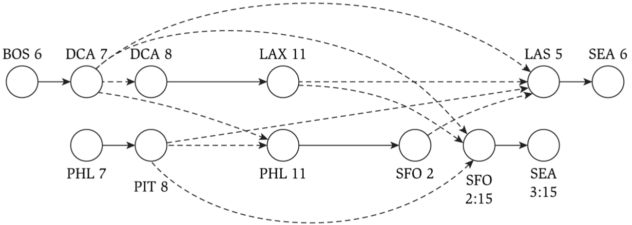


- Desire to serve m specific flight segments.
- Each flight segment (or flight) specified by four parameters: origin airport, destination airport, departure time, arrival time.
- We can use a single plane for flight i and later for flight j if
  - (i) the destination of i is the same as the origin of j and there is enough time to perform maintenance on the plane between the two flights, or
  - (ii) we can add a flight that takes the plane from the destination of i to the origin of j with enough time for maintenance.
- Goal is to schedule all m flights using at most k planes.

Reachability



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- Flight j is *reachable* from flight i if the same plane can be used for both flights subject to the constraints described earlier.
- Assume input includes pairs (i, j) of reachable flights, i.e., in each pair j is reachable from i.
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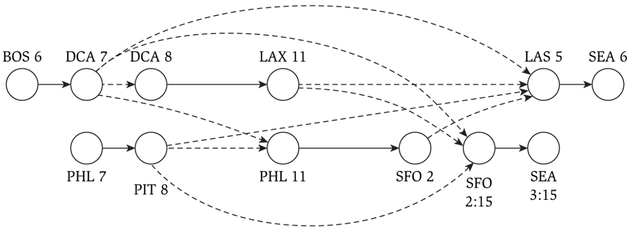

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- Flight j is *reachable* from flight i if the same plane can be used for both flights subject to the constraints described earlier.
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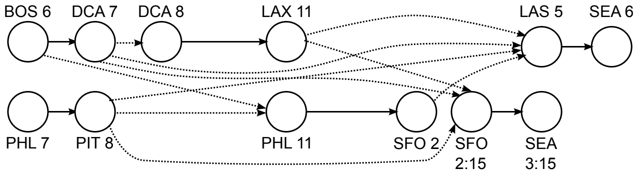
  - Pairs form a DAG.
  - *Flights* are reachable from one another, not *airports*.
  - Construction of reachable pairs will take maintenance time into account.
  - Definition of reachability can be more complex; input pairs can encode this complexity.

The Airline Scheduling Problem



Airline Scheduling

INSTANCE: Set S of m flight segments (u_i, v_i), $1 \leq i \leq m$, a set R of reachable pairs of flights (i, j), $1 \leq i, j \leq m$, and an integer bound k

SOLUTION: *Feasible scheduling*:

(a) Set T of $n \geq 0$ new flight segments (u_j, v_j), $1 \leq j \leq n$ and

(b) A partition of $S \cup T$ into at most k sequences such that in each sequence, flight i is reachable from flight $i-1$, for all $1 < i \leq l$, where l is the length of the sequence.

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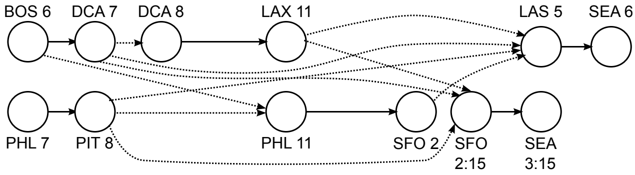

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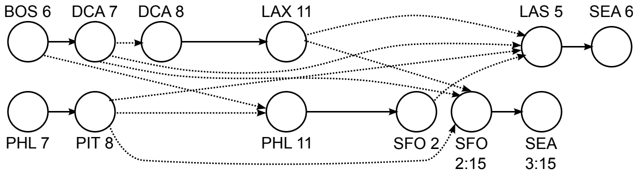

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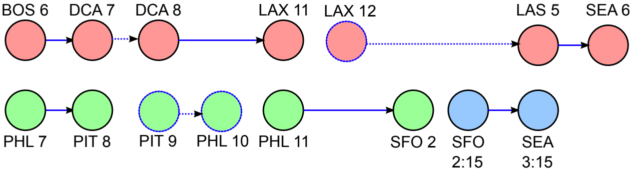

The dotted circles are meant only to illustrate the new flights added.

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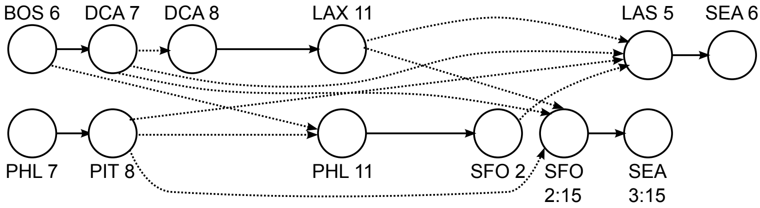

- Nodes in the flow network are airports.
- Planes correspond to units of flow.

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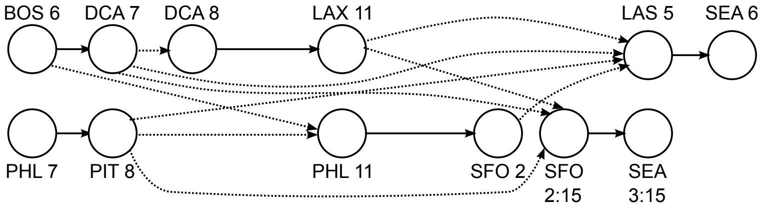

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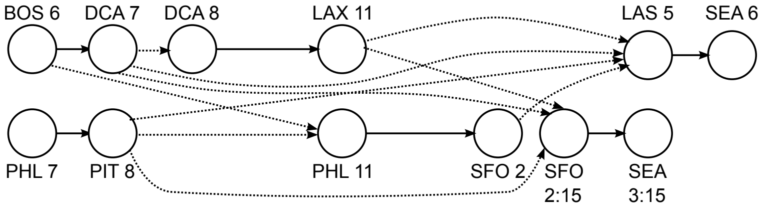

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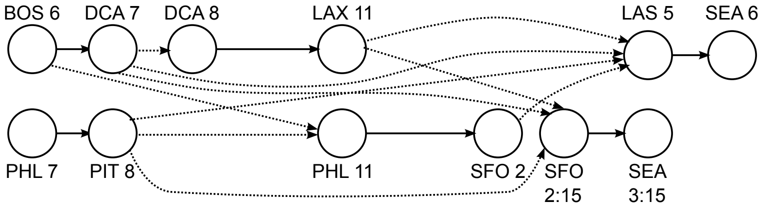

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- Each flight corresponds to an edge. How do we ensure each flight is served by exactly one plane? Lower bound of 1 and a capacity of 1.
- How do we represent reachability? If (i, j) is a reachable pair, there is an edge from v_i to u_j with lower bound of 0 and a capacity of 1.

Designing the Flow Network

Nodes:
- For each flight i, graph G has two nodes u_i and v_i.
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Same plane for flights i and j: For each $(i, j) \in R$, G contains an edge directed from v_i to u_j with a lower bound of 0 and a capacity of 1.

Start a plane with any flight: For each $i \in S$, G contains an edge directed from s to u_i with a lower bound of 0 and a capacity of 1.

End a plane with any flight: For each $j \in S$, G contains an edge directed from v_j to t with a lower bound of 0 and a capacity of 1.

Excess planes: G contains an edge directed from s to t with lower bound 0 and capacity k.

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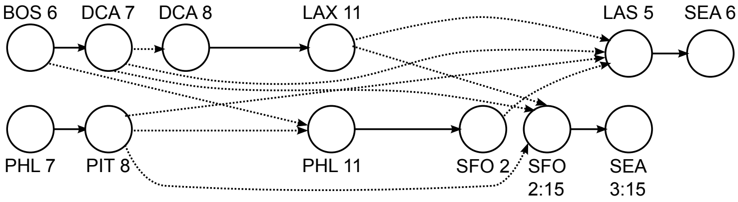
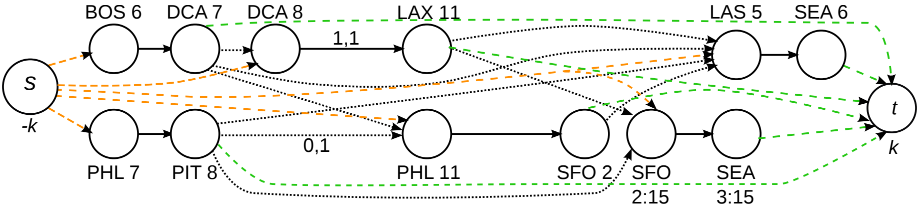
Goal: Compute whether G has a feasible circulation.

Example of Circulation Formulation





The image does not show the edge between s and t.

Proof of Correctness: Part 1

- Claim: We can schedule all flights in S using at most k planes iff G has a feasible circulation.

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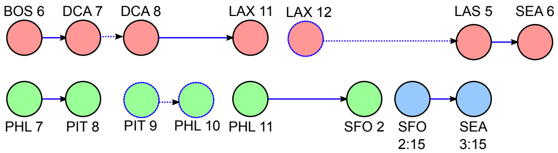

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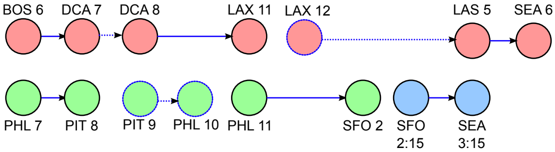

- Claim: We can schedule all flights in S using at most k planes iff G has a feasible circulation.
- Feasible schedule with $k' \leq k$ planes $\Rightarrow$ feasible circulation:
  - Each plane $l, 1 \leq l \leq k'$ flies along a particular path P_l of flights unique to that plane, starting at city s_l and ending at city t_l.
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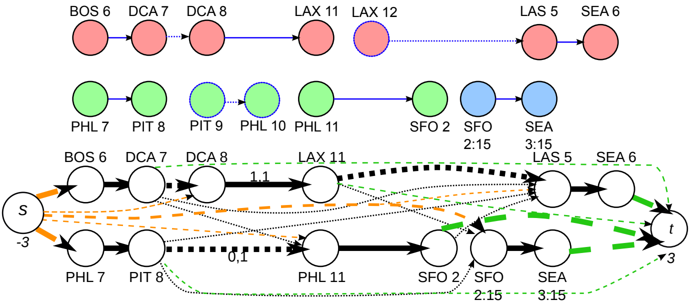

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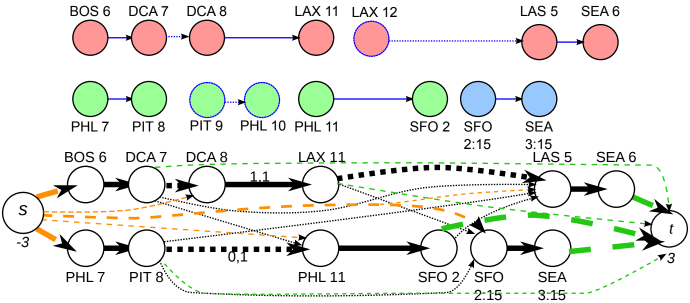

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  - Why does the resulting circulation satisfy all demand, lower bound, and capacity constraints?

Proof of Correctness: Part 2

- Claim: We can schedule all flights in S using at most k planes iff G has a feasible circulation.

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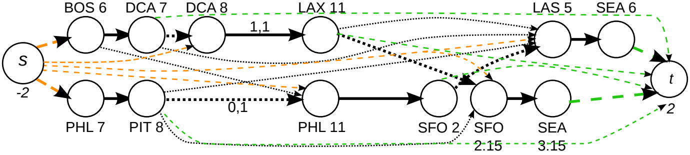

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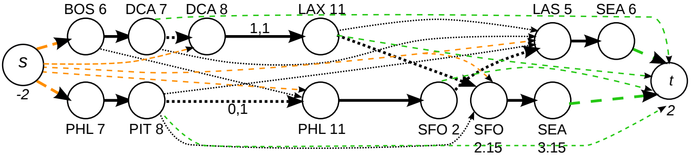

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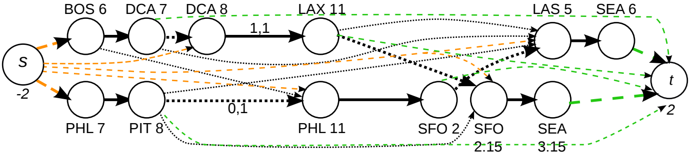

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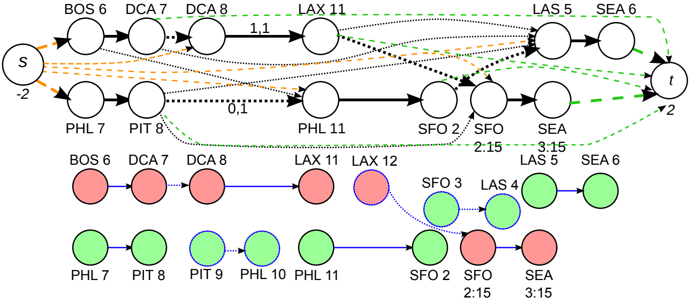

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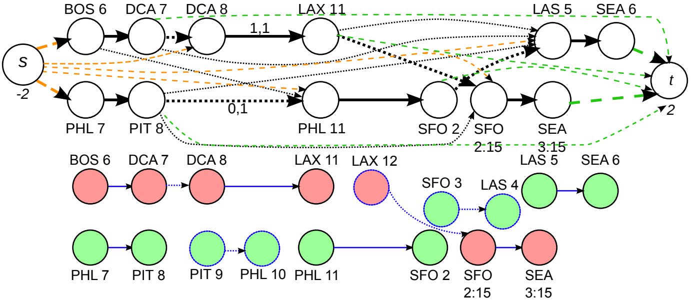

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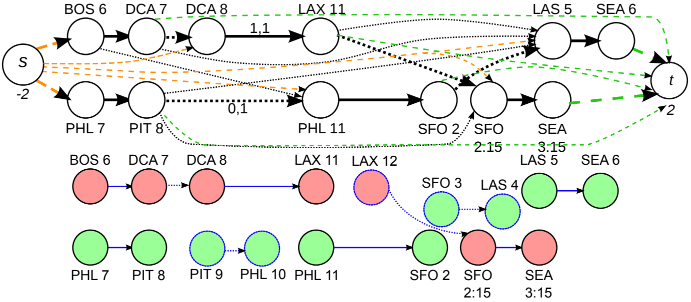

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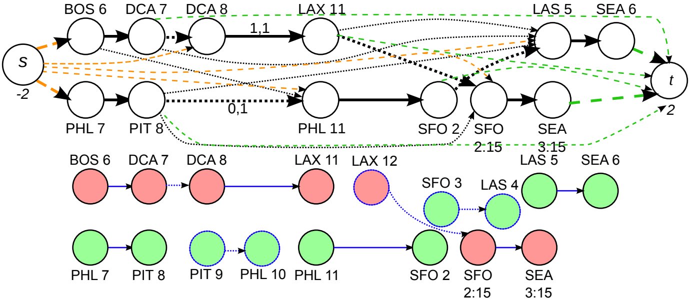

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  - Output these paths. Paths define extra flight segments automatically.