Dynamic Programming

T. M. Murali

March 17, 22, 24, 2016
Algorithm Design Techniques

1. Goal: design efficient (polynomial-time) algorithms.
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   ▶ Pro: natural approach to algorithm design.
   ▶ Con: many greedy approaches to a problem. Only some may work.
   ▶ Con: many problems for which no greedy approach is known.
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   - Pro: simple to develop algorithm skeleton.
   - Con: conquer step can be very hard to implement efficiently.
   - Con: usually reduces time for a problem known to be solvable in polynomial time.
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   - **Pro:** simple to develop algorithm skeleton.
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4. Dynamic programming
   - More powerful than greedy and divide-and-conquer strategies.
   - *Implicitly* explore space of all possible solutions.
   - Solve multiple sub-problems and build up correct solutions to larger and larger sub-problems.
   - Careful analysis needed to ensure number of sub-problems solved is polynomial in the size of the input.
History of Dynamic Programming

- Bellman pioneered the systematic study of dynamic programming in the 1950s.
History of Dynamic Programming

- Bellman pioneered the systematic study of dynamic programming in the 1950s.
- The Secretary of Defense at that time was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - “it’s impossible to use dynamic in a pejorative sense”
  - “something not even a Congressman could object to” (Bellman, R. E., Eye of the Hurricane, An Autobiography).
Applications of Dynamic Programming

- Computational biology: Smith-Waterman algorithm for sequence alignment.
- Operations research: Bellman-Ford algorithm for shortest path routing in networks.
- Control theory: Viterbi algorithm for hidden Markov models.
- Computer science (theory, graphics, AI, ...): Unix `diff` command for comparing two files.
Review: Interval Scheduling

**Interval Scheduling**

**INSTANCE:** Nonempty set \(\{(s_i, f_i), 1 \leq i \leq n\}\) of start and finish times of \(n\) jobs.

**SOLUTION:** The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
Review: Interval Scheduling

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SOLUTION: The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- Greedy algorithm: sort jobs in increasing order of finish times. Add next job to current subset only if it is compatible with previously-selected jobs.
**Weighted Interval Scheduling**

**INSTANCE:** Nonempty set \( \{(s_i, f_i), 1 \leq i \leq n\} \) of start and finish times of \( n \) jobs and a weight \( v_i \geq 0 \) associated with each job.

**SOLUTION:** A set \( S \) of mutually compatible jobs such that \( \sum_{i \in S} v_i \) is maximised.

![Diagram](https://via.placeholder.com/150)

*Figure 6.1* A simple instance of weighted interval scheduling.
Weighted Interval Scheduling

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<table>
<thead>
<tr>
<th>Index</th>
<th>Value</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

*Figure 6.1* A simple instance of weighted interval scheduling.

- Greedy algorithm can produce arbitrarily bad results for this problem.
Detour: a Binomial Identity

Pascal's triangle:
- Each element is a binomial coefficient.
- Each element is the sum of the two elements above it.

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

Proof: either we include the \(n\)th element in a subset or not...
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Approach

- Sort jobs in increasing order of finish time and relabel: $f_1 \leq f_2 \leq \ldots \leq f_n$.
- Job $i$ comes before job $j$ if $i < j$.
- $p(j)$ is the largest index $i < j$ such that job $i$ is compatible with job $j$. $p(j) = 0$ if there is no such job $i$.
- All jobs that come before job $p(j)$ are also compatible with job $j$.

![Diagram](image)

**Figure 6.2** An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.

- We will develop optimal algorithm from obvious statements about the problem.
Sub-problems

Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

Case 1 job $n$ is not in $O$.

Case 2 job $n$ is in $O$. 
Sub-problems

Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

Case 1 job $n$ is not in $O$. $O$ must be the optimal solution for jobs $\{1, 2, \ldots, n - 1\}$.

Case 2 job $n$ is in $O$. 

$O$ cannot use incompatible jobs $\{p(n) + 1, p(n) + 2, \ldots, n - 1\}$.

Remaining jobs in $O$ must be the optimal solution for jobs $\{1, 2, \ldots, p(n)\}$.

$O$ must be the best of these two choices!
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- **Case 2** job $n$ is in $O$.
  - $O$ cannot use incompatible jobs \{\(p(n) + 1, p(n) + 2, \ldots, n - 1\}\).
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Sub-problems

Let $\mathcal{O}$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

Case 1 job $n$ is not in $\mathcal{O}$. $\mathcal{O}$ must be the optimal solution for jobs $\{1, 2, \ldots, n-1\}$.

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- $\mathcal{O}$ cannot use incompatible jobs $\{p(n) + 1, p(n) + 2, \ldots, n-1\}$.
- Remaining jobs in $\mathcal{O}$ must be the optimal solution for jobs $\{1, 2, \ldots, p(n)\}$.

$\mathcal{O}$ must be the best of these two choices!

Suggests finding optimal solution for sub-problems consisting of jobs $\{1, 2, \ldots, j-1, j\}$, for all values of $j$. 

T. M. Murali  March 17, 22, 24, 2016  Dynamic Programming
Recursion

Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).
Recursion

- Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).
- We are seeking $O_n$ with a value of $OPT(n)$. 

When does request $j$ belong to $O_j$?
- If and only if $v_j + OPT(p(j)) \geq OPT(j-1)$. 

T. M. Murali March 17, 22, 24, 2016 Dynamic Programming
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- To compute $OPT(j)$:
  - Case 1: $j \notin O_j$:
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- To compute $OPT(j)$:
  - Case 1 $j \not\in O_j$: $OPT(j) = OPT(j - 1)$.
  - Case 2 $j \in O_j$: $OPT(j) = v_j + OPT(p(j))$
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Case 1 $j \notin O_j$: $OPT(j) = OPT(j - 1)$.

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$OPT(j) = \max(v_j + OPT(p(j)), OPT(j - 1))$
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$$OPT(j) = \max(v_j + OPT(p(j)), OPT(j - 1))$$

When does request $j$ belong to $O_j$? If and only if $v_j + OPT(p(j)) \geq OPT(j - 1)$. 

T. M. Murali March 17, 22, 24, 2016 Dynamic Programming
Recursive Algorithm

```
Compute-Opt(j)
    If j = 0 then
        Return 0
    Else
        Return max(v_j + Compute-Opt(p(j)), Compute-Opt(j - 1))
    Endif
```
Recursive Algorithm

Compute-Opt(j)

If \( j = 0 \) then
   Return 0

Else
   Return \( \max(v_j + \text{Compute-Opt}(p(j)), \text{Compute-Opt}(j - 1)) \)
Endif

- Correctness of algorithm follows by induction (see textbook for proof).
Example of Recursive Algorithm

Index

1 \hspace{1cm} v_1 = 2 \hspace{1cm} p(1) = 0
2 \hspace{1cm} v_2 = 4 \hspace{1cm} p(2) = 0
3 \hspace{1cm} v_3 = 4 \hspace{1cm} p(3) = 1
4 \hspace{1cm} v_4 = 7 \hspace{1cm} p(4) = 0
5 \hspace{1cm} v_5 = 2 \hspace{1cm} p(5) = 3
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Figure 6.2 An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).

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\begin{align*}
\text{OPT}(6) &= \\
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Optimal solution is job 5, job 3, and job 1.
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Figure 6.2 An instance of weighted interval scheduling with the functions \(p(j)\) defined for each interval \(j\).
**Example of Recursive Algorithm**

![Diagram](image)

**Figure 6.2** An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.

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Example of Recursive Algorithm

Optimal solution is job 5, job 3, and job 1.

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**Example of Recursive Algorithm**

![Diagram showing the recursive algorithm](image)

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\text{OPT}(4) &= \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) = 7 \\
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\text{OPT}(2) &= \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \\
\text{OPT}(1) &= v_1 = 2 \\
\text{OPT}(0) &= 0
\end{align*}
\]

The optimal solution is job 5, job 3, and job 1.
Example of Recursive Algorithm

\[ \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) = 8 \]
\[ \text{OPT}(5) = \max(v_5 + \text{OPT}(p(5)), \text{OPT}(4)) = \max(2 + \text{OPT}(3), \text{OPT}(4)) = 8 \]
\[ \text{OPT}(4) = \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) = 7 \]
\[ \text{OPT}(3) = \max(v_3 + \text{OPT}(p(3)), \text{OPT}(2)) = \max(4 + \text{OPT}(1), \text{OPT}(2)) = 6 \]
\[ \text{OPT}(2) = \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \]
\[ \text{OPT}(1) = v_1 = 2 \]
\[ \text{OPT}(0) = 0 \]

Optimal solution is job 5, job 3, and job 1.

**Figure 6.2** An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).
Running Time of Recursive Algorithm

Compute-Opt(j)

If j = 0 then
  Return 0
Else
  Return max(\(v_j+\text{Compute-Opt}(p(j))\), \(\text{Compute-Opt}(j-1)\))
Endif
Running Time of Recursive Algorithm

What is the running time of the algorithm?

```plaintext
Compute-Opt(j)
    If j = 0 then
        Return 0
    Else
        Return max(v_j + Compute-Opt(p(j)), Compute-Opt(j - 1))
    Endif
```

Can be exponential in \( n \). When \( p(j) = j - 2 \), for all \( j \geq 2 \): recursive calls are for \( j - 1 \) and \( j - 2 \).
Running Time of Recursive Algorithm

Compute-Opt(j)
If \( j = 0 \) then
    Return 0
Else
    Return max(\( v_j + \text{Compute-Opt}(p(j)) \), \( \text{Compute-Opt}(j - 1) \))
Endif

What is the running time of the algorithm? Can be exponential in \( n \).
Running Time of Recursive Algorithm

What is the running time of the algorithm? Can be exponential in $n$.

When $p(j) = j - 2$, for all $j \geq 2$: recursive calls are for $j - 1$ and $j - 2$.

---

Compute-Opt($j$)
If $j = 0$ then
    Return 0
Else
    Return max($v_j$ + Compute-Opt($p(j)$), Compute-Opt($j - 1$))
Endif

---

**Figure 6.4** An instance of weighted interval scheduling on which the simple Compute-Opt recursion will take exponential time. The values of all intervals in this instance are 1.

**Figure 6.3** The tree of subproblems called by Compute-Opt on the problem instance of Figure 6.2.
Memoisation

- Store $OPT(j)$ values in a cache and reuse them rather than recompute them.
Memoisation

- Store $\text{OPT}(j)$ values in a cache and reuse them rather than recompute them.

M-Compute-Opt($j$)

If $j = 0$ then
   Return 0
Else if $M[j]$ is not empty then
   Return $M[j]$ Else
   Define $M[j] = \max(v_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j - 1))$
   Return $M[j]$
Endif
Running Time of Memoisation

\[
\text{M-Compute-Opt}(j) \\
\text{If } j = 0 \text{ then} \\
\quad \text{Return 0} \\
\text{Else if } M[j] \text{ is not empty then} \\
\quad \text{Return } M[j] \\
\text{Else} \\
\quad \text{Define } M[j] = \max(v_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j - 1)) \\
\quad \text{Return } M[j] \\
\text{Endif}
\]

Claim: running time of this algorithm is \(O(n)\) (after sorting).
Running Time of Memoisation

M-Compute-Opt(j)
    If \( j = 0 \) then
        Return 0
    Else if \( M[j] \) is not empty then
        Return \( M[j] \)
    Else
        Define \( M[j] = \max(v_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j - 1)) \)
        Return \( M[j] \)
    Endif

Claim: running time of this algorithm is \( O(n) \) (after sorting).
Time spent in a single call to M-Compute-Opt is \( O(1) \) apart from time spent in recursive calls.
Total time spent is the order of the number of recursive calls to M-Compute-Opt.
How many such recursive calls are there in total?
**Running Time of Memoisation**

M-Compute-Opt\( (j) \)

If \( j = 0 \) then
- Return 0
Else if \( M[j] \) is not empty then
- Return \( M[j] \)
Else
- Define \( M[j] = \max (v_j + M-\text{Compute-Opt}(p(j)), M-\text{Compute-Opt}(j - 1)) \)
- Return \( M[j] \)
Endif

- **Claim:** running time of this algorithm is \( O(n) \) (after sorting).
- **Time spent in a single call to M-Compute-Opt is \( O(1) \) apart from time spent in recursive calls.
- **Total time spent is the order of the number of recursive calls to M-Compute-Opt.**
- **How many such recursive calls are there in total?**
- **Use number of filled entries in \( M \) as a measure of progress.**
- **Each time M-Compute-Opt issues two recursive calls, it fills in a new entry in \( M \).**
- **Therefore, total number of recursive calls is \( O(n) \).**
Computing $O$ in Addition to $\text{OPT}(n)$
Computing $O$ in Addition to $\text{OPT}(n)$

- Explicitly store $O_j$ in addition to $\text{OPT}(j)$.
Computing $\mathcal{O}$ in Addition to $\text{OPT}(n)$

- Explicitly store $\mathcal{O}_j$ in addition to $\text{OPT}(j)$. Running time becomes $O(n^2)$. 

Recall: request $j$ belong to $\mathcal{O}_j$ if and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j - 1)$.

Can recover $\mathcal{O}_j$ from values of the optimal solutions in $\mathcal{O}(j)$ time.
Computing $O$ in Addition to $OPT(n)$

- Explicitly store $O_j$ in addition to $OPT(j)$. Running time becomes $O(n^2)$.
- Recall: request $j$ belong to $O_j$ if and only if $v_j + OPT(p(j)) \geq OPT(j - 1)$.
- Can recover $O_j$ from values of the optimal solutions in $O(j)$ time.
Computing $O$ in Addition to $OPT(n)$

- Explicitly store $O_j$ in addition to $OPT(j)$. Running time becomes $O(n^2)$.
- Recall: request $j$ belong to $O_j$ if and only if $v_j + OPT(p(j)) \geq OPT(j - 1)$.
- Can recover $O_j$ from values of the optimal solutions in $O(j)$ time.

---

```
Find-Solution(j)
If $j = 0$ then
    Output nothing
Else
    If $v_j + M[p(j)] \geq M[j - 1]$ then
        Output $j$ together with the result of Find-Solution($p(j)$)
    Else
        Output the result of Find-Solution($j - 1$)
Endif
Endif
```
From Recursion to Iteration

- Unwind the recursion and convert it into iteration.
- Can compute values in \( M \) iteratively in \( O(n) \) time.
- Find-Solution works as before.

Iterative-Compute-Opt

\[
\begin{align*}
 M[0] &= 0 \\
 \text{For } j = 1, 2, \ldots, n \\
 M[j] &= \max(v_j + M[p(j)], M[j - 1]) \\
 \text{Endfor}
\end{align*}
\]
Basic Outline of Dynamic Programming

To solve a problem, we need a collection of sub-problems that satisfy a few properties:

1. There are a polynomial number of sub-problems.
2. The solution to the problem can be computed easily from the solutions to the sub-problems.
3. There is a natural ordering of the sub-problems from “smallest” to “largest”.
4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.
Basic Outline of Dynamic Programming

To solve a problem, we need a collection of sub-problems that satisfy a few properties:

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3. There is a natural ordering of the sub-problems from “smallest” to “largest”.
4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.

Difficulties in designing dynamic programming algorithms:

1. Which sub-problems to define?
2. How can we tie together sub-problems using a recurrence?
3. How do we order the sub-problems (to allow iterative computation of optimal solutions to sub-problems)?
Imagery from new street view vehicles is accompanied by laser range data, which is aggregated and simplified by robustly fitting it in a coarse mesh that models the dominant scene surfaces.
Imagery from new street view vehicles is accompanied by laser range data, which is aggregated and simplified by robustly fitting it in a coarse mesh that models the dominant scene surfaces.
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Fitting Lines
Fitting Lines
Fitting Lines

![Fitting Lines Diagram]
Fitting Lines
Fitting Lines
Least Squares Problem

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.

Figure 6.6 A “line of best fit.”

Least Squares INSTANCE:
Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

SOLUTION:
Line $L$:

\[
y = ax + b
\]

that minimises Error($L$, $P$) =

\[
\sum_{i=1}^{n} (y_i - ax_i - b)^2
\]

Minimisation is over all possible choices of $a$ and $b$.

Solution is achieved by

\[
a = \frac{n \sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

and

\[
b = \frac{\sum_{i=1}^{n} y_i - a \sum_{i=1}^{n} x_i}{n}
\]
Least Squares Problem

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.
- How do we formalise the problem?

Figure 6.6 A “line of best fit.”
Given scientific or statistical data plotted on two axes.

Find the “best” line that “passes” through these points.

How do we formalise the problem?

**Least Squares**

**INSTANCE:** Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

**SOLUTION:** Line $L : y = ax + b$ that minimises

$$
\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.
$$
Least Squares Problem

- Given scientific or statistical data plotted on two axes.
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Least Squares

**INSTANCE:** Set \( P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \) of \( n \) points.

**SOLUTION:** Line \( L : y = ax + b \) that minimises

\[
\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.
\]

- Minimisation is over all possible choices of

![Figure 6.6 A “line of best fit.”](image-url)


**Least Squares Problem**

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.
- How do we formalise the problem?

---

**Least Squares**

**INSTANCE:** Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

**SOLUTION:** Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$ 

- Minimisation is over all possible choices of $a$ and $b$.
- Solution is achieved by

\[
a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2} \quad \text{and} \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}
\]
Segmented Least Squares

Figure 6.7 A set of points that lie approximately on two lines.
**Segmented Least Squares**

*Figure 6.7* A set of points that lie approximately on two lines. *Figure 6.8* A set of points that lie approximately on three lines.
Segmented Least Squares

Figure 6.7 A set of points that lie approximately on two lines.

Figure 6.8 A set of points that lie approximately on three lines.

- Want to fit multiple lines through $P$.
- Each line must fit contiguous set of $x$-coordinates.
- Lines must minimise total error.
Example of Segmented Least Squares

Input contains a set of two-dimensional points.
Example of Segmented Least Squares

Consider the $x$-coordinates of the points in the input.
Divide the points into segments; each segment contains consecutive points in the sorted order by $x$-coordinate.
Example of Segmented Least Squares

Fit the best line for each segment.
Example of Segmented Least Squares

Illegal solution: black point is not in any segment.
Example of Segmented Least Squares

Illegal solution: leftmost purple point has $x$-coordinate between last two points in green segment.
Segmented Least Squares

Figure 6.7 A set of points that lie approximately on two lines.

Figure 6.8 A set of points that lie approximately on three lines.
**Segmented Least Squares**

**INSTANCE:** Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of $n$ points, $x_1 < x_2 < \cdots < x_n$.

**SOLUTION:** A integer $k$, a partition of $P$ into $k$ segments $\{P_1, P_2, \ldots, P_k\}$, $k$ lines $L_j : y = a_j x + b_j, 1 \leq j \leq k$ that minimise

$$\sum_{j=1}^{k} \text{Error}(L_j, P_j)$$

- A subset $P'$ of $P$ is a *segment* if $1 \leq i < j \leq n$ exist such that $P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\}$. 

*Figure 6.7* A set of points that lie approximately on two lines.  *Figure 6.8* A set of points that lie approximately on three lines.
Segmented Least Squares

**INSTANCE:** Set \( P = \{p_i = (x_i, y_i), 1 \leq i \leq n\} \) of \( n \) points, \( x_1 < x_2 < \cdots < x_n \) and a parameter \( C > 0 \).

**SOLUTION:** A integer \( k \), a partition of \( P \) into \( k \) segments \( \{P_1, P_2, \ldots, P_k\} \), \( k \) lines \( L_j : y = a_j x + b_j, 1 \leq j \leq k \) that minimise

\[
\sum_{j=1}^{k} \text{Error}(L_j, P_j) + Ck
\]

- A subset \( P' \) of \( P \) is a *segment* if \( 1 \leq i < j \leq n \) exist such that \( P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\} \).
Observation: $p_n$ is part of some segment in the optimal solution. This segment starts at some point $p_i$.

Let $OPT(i)$ be the optimal value for the points $\{p_1, p_2, \ldots, p_i\}$.

Let $e_{i,j}$ denote the minimum error of a (single) line that fits $\{p_i, p_2, \ldots, p_j\}$.

We want to compute $OPT(n)$.

If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

$$OPT(n) = e_{i,n} + C + OPT(i - 1)$$
Formulating the Recursion II

Consider the sub-problem on the points \( \{ p_1, p_2, \ldots, p_j \} \)

To obtain \( \text{OPT}(j) \), if the last segment in the optimal partition is \( \{ p_i, p_{i+1}, \ldots, p_j \} \), then

\[
\text{OPT}(j) = e_{i,j} + C + \text{OPT}(i - 1)
\]
Formulating the Recursion II

Consider the sub-problem on the points \{p_1, p_2, \ldots p_j\}

To obtain \text{OPT}(j), if the last segment in the optimal partition is \{p_i, p_{i+1}, \ldots, p_j\}, then

\[
\text{OPT}(j) = e_{i,j} + C + \text{OPT}(i - 1)
\]

Since \(i\) can take only \(j\) distinct values,

\[
\text{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \text{OPT}(i - 1))
\]

Segment \{p_i, p_{i+1}, \ldots p_j\} is part of the optimal solution for this sub-problem if and only if the minimum value of \text{OPT}(j) is obtained using index \(i\).
Dynamic Programming Algorithm

\[ \text{OPT}(j) = \min_{1 \leq i \leq j} \left( e_{i,j} + C + \text{OPT}(i - 1) \right) \]

Segmented-Least-Squares(n)

Array \( M[0...n] \)

Set \( M[0] = 0 \)

For all pairs \( i \leq j \)

\[
\text{Compute the least squares error } e_{i,j} \text{ for the segment } p_i,\ldots,p_j
\]

Endfor

For \( j = 1, 2, \ldots, n \)

Use the recurrence (6.7) to compute \( M[j] \)

Endfor

Return \( M[n] \)
Dynamic Programming Algorithm

\[
\text{OPT}(j) = \min_{1 \leq i \leq j} \left( e_{i,j} + C + \text{OPT}(i - 1) \right)
\]

Segmented-Least-Squares(n)

Array \( M[0 \ldots n] \)
Set \( M[0] = 0 \)
For all pairs \( i \leq j \)
\hspace{1em} Compute the least squares error \( e_{i,j} \) for the segment \( p_i, \ldots, p_j \)
Endfor
For \( j = 1, 2, \ldots, n \)
\hspace{1em} Use the recurrence (6.7) to compute \( M[j] \)
Endfor
Return \( M[n] \)

- Running time is \( O(n^3) \), can be improved to \( O(n^2) \).
- We can find the segments in the optimal solution by backtracking.
RNA Molecules

- RNA is a basic biological molecule. It is single stranded.
- RNA molecules fold into complex “secondary structures.”
- Secondary structure often governs the behaviour of an RNA molecule.
- Various rules govern secondary structure formation:
RNA Molecules

- RNA is a basic biological molecule. It is single stranded.
- RNA molecules fold into complex “secondary structures.”
- Secondary structure often governs the behaviour of an RNA molecule.
- Various rules govern secondary structure formation:
  1. Pairs of bases match up; each base matches with \( \leq 1 \) other base.
  2. Adenine always matches with Uracil.
  3. Cytosine always matches with Guanine.
  4. There are no kinks in the folded molecule.
  5. Structures are “knot-free”.

Figure 6.13 An RNA secondary structure. Thick lines connect adjacent elements of the sequence; thin lines indicate pairs of elements that are matched.
RNA Molecules

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RNA molecules fold into complex “secondary structures.”
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Problem: given an RNA molecule, predict its secondary structure.
RNA Molecules

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  3. Cytosine always matches with Guanine.
  4. There are no kinks in the folded molecule.
  5. Structures are “knot-free”.

Problem: given an RNA molecule, predict its secondary structure.
Hypothesis: In the cell, RNA molecules form the secondary structure with the lowest total free energy.

Figure 6.13 An RNA secondary structure. Thick lines connect adjacent elements of the sequence; thin lines indicate pairs of elements that are matched.
Formulating the Problem

- An RNA molecule is a string $B = b_1 b_2 \ldots b_n$; each $b_i \in \{A, C, G, U\}$.
- A secondary structure on $B$ is a set of pairs $S = \{(i, j)\}$, where $1 \leq i, j \leq n$ and

Figure 6.14 Two views of an RNA secondary structure. In the second view, (b), the string has been "stretched" lengthwise, and edges connecting matched pairs appear as noncrossing "bubbles" over the string.
An **RNA molecule** is a string \( B = b_1 b_2 \ldots b_n \); each \( b_i \in \{A, C, G, U\} \).

A **secondary structure on** \( B \) is a set of pairs \( S = \{(i, j)\} \), where \( 1 \leq i, j \leq n \) and

1. (**No kinks.**) If \( (i, j) \in S \), then \( i < j - 4 \).
2. (**Watson-Crick**) The elements in each pair in \( S \) consist of either \( \{A, U\} \) or \( \{C, G\} \) (in either order).
3. \( S \) is a **matching**: no index appears in more than one pair.
4. (**No knots**) If \( (i, j) \) and \( (k, l) \) are two pairs in \( S \), then we cannot have \( i < k < j < l \).

The **energy** of a secondary structure \( \propto \) the number of base pairs in it.

Problem: Compute the largest secondary structure, i.e., with the largest number of base pairs.
Illegal Secondary Structures

A C A U G G C C A U G U

Watson-Crick

Kink Matching

Knot

T. M. Murali  March 17, 22, 24, 2016 Dynamic Programming
Legal Secondary Structures

A C A U G G C C A U G U

A C A U G G C C A U G U

A C A U G G C C A U G U

A C A U G G C C A U G U

A C A U G G C C A U G U

A C A U G G C C A U G U

A C A U G G C C A U G U
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. 

Insight: need sub-problems indexed both by start and by end.
Dynamic Programming Approach

- \( OPT(j) \) is the maximum number of base pairs in a secondary structure for \( b_1 b_2 \ldots b_j \). \( OPT(j) = 0 \), if \( j \leq 5 \).
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.
- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.
- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$. 
Dynamic Programming Approach

- \( OPT(j) \) is the maximum number of base pairs in a secondary structure for \( b_1 b_2 \ldots b_j \). \( OPT(j) = 0 \), if \( j \leq 5 \).

- In the optimal secondary structure on \( b_1 b_2 \ldots b_j \)
  1. if \( j \) is not a member of any pair, use \( OPT(j - 1) \).
  2. if \( j \) pairs with some \( t < j - 4 \),

\[ \text{Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.} \]
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.
- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, knot condition yields two independent sub-problems!

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Dynamic Programming Approach

- \( OPT(j) \) is the maximum number of base pairs in a secondary structure for \( b_1 b_2 \ldots b_j \). \( OPT(j) = 0 \), if \( j \leq 5 \).
- In the optimal secondary structure on \( b_1 b_2 \ldots b_j \):
  1. if \( j \) is not a member of any pair, use \( OPT(j - 1) \).
  2. if \( j \) pairs with some \( t < j - 4 \), knot condition yields two independent sub-problems! \( OPT(t - 1) \) and ???

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.

- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, knot condition yields two independent sub-problems! $OPT(t - 1)$ and ??

- Insight: need sub-problems indexed both by start and by end.

![Diagram](image)

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

OPT(i, j) is the maximum number of base pairs in a secondary structure for $b_ib_{i+1} \ldots b_j$.

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

$\text{OPT}(i, j)$ is the maximum number of base pairs in a secondary structure for $b_ib_{i+1}\ldots b_j$. $\text{OPT}(i, j) = 0$, if $i \geq j - 4$. 

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

▶ \( OPT(i, j) \) is the maximum number of base pairs in a secondary structure for \( b_i b_{i+1} \ldots b_j \). \( OPT(i, j) = 0, \) if \( i \geq j - 4 \).

▶ In the optimal secondary structure on \( b_i b_{i+2} \ldots b_j \)

\[
OPT(i, j) = \max \left( \right)
\]

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
**Correct Dynamic Programming Approach**

- \( \text{OPT}(i, j) \) is the maximum number of base pairs in a secondary structure for \( b_i b_{i+1} \ldots b_j \). \( \text{OPT}(i, j) = 0 \), if \( i \geq j - 4 \).

- In the optimal secondary structure on \( b_i b_{i+2} \ldots b_j \):
  1. if \( j \) is not a member of any pair, compute \( \text{OPT}(i, j - 1) \).

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \right.

\left. \right)
\]
**Correct Dynamic Programming Approach**

- \( \text{OPT}(i, j) \) is the maximum number of base pairs in a secondary structure for \( b_ib_{i+1} \ldots b_j \). \( \text{OPT}(i, j) = 0, \) if \( i \geq j - 4 \).
- In the optimal secondary structure on \( b_ib_{i+2} \ldots b_j \)
  1. if \( j \) is not a member of any pair, compute \( \text{OPT}(i, j - 1) \).
  2. if \( j \) pairs with some \( t < j - 4 \), compute \( \text{OPT}(i, t - 1) \) and \( \text{OPT}(t + 1, j - 1) \).

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \right.
\]

---

*Figure 6.15* Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

\[ \text{OPT}(i, j) \] is the maximum number of base pairs in a secondary structure for \( b_i b_{i+1} \ldots b_j \). \( \text{OPT}(i, j) = 0 \), if \( i \geq j - 4 \).

\[ \text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_{t < j - 4} \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right) \]

- In the optimal secondary structure on \( b_i b_{i+2} \ldots b_j \)
  1. if \( j \) is not a member of any pair, compute \( \text{OPT}(i, j - 1) \).
  2. if \( j \) pairs with some \( t < j - 4 \), compute \( \text{OPT}(i, t - 1) \) and \( \text{OPT}(t + 1, j - 1) \).

Since \( t \) can range from \( i \) to \( j - 5 \),

\[ \text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_{t < j - 4} \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right) \]

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

- $OPT(i, j)$ is the maximum number of base pairs in a secondary structure for $b_i b_{i+1} \ldots b_j$. $OPT(i, j) = 0$, if $i \geq j - 4$.
- In the optimal secondary structure on $b_i b_{i+2} \ldots b_j$
  1. if $j$ is not a member of any pair, compute $OPT(i, j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, compute $OPT(i, t - 1)$ and $OPT(t + 1, j - 1)$.
- Since $t$ can range from $i$ to $j - 5$,

$$OPT(i, j) = \max \left( OPT(i, j - 1), \max_t \left(1 + OPT(i, t - 1) + OPT(t + 1, j - 1)\right) \right)$$
Correct Dynamic Programming Approach

▶ \( \text{OPT}(i, j) \) is the maximum number of base pairs in a secondary structure for \( b_i b_{i+1} \ldots b_j \). \( \text{OPT}(i, j) = 0 \), if \( i \geq j - 4 \).

▶ In the optimal secondary structure on \( b_i b_{i+2} \ldots b_j \):

1. if \( j \) is not a member of any pair, compute \( \text{OPT}(i, j - 1) \).
2. if \( j \) pairs with some \( t < j - 4 \), compute \( \text{OPT}(i, t - 1) \) and \( \text{OPT}(t + 1, j - 1) \).

▶ Since \( t \) can range from \( i \) to \( j - 5 \),

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_t \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right)
\]

▶ In the “inner” maximisation, \( t \) runs over all indices between \( i \) and \( j - 5 \) that are allowed to pair with \( j \).

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Example of Dynamic Programming Algorithm

A C A U G G C C A U G U
A C A U G G C C A U G U
A C A U G G C C A U G

T. M. Murali March 17, 22, 24, 2016 Dynamic Programming
Dynamic Programming Algorithm

\[ \text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_{t} (1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1)) \right) \]

- There are \( O(n^2) \) sub-problems.
- How do we order them from “smallest” to “largest”?
Dynamic Programming Algorithm

\[ \text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_t \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right) \]

- There are \( O(n^2) \) sub-problems.
- How do we order them from “smallest” to “largest”?
- Note that computing \( \text{OPT}(i, j) \) involves sub-problems \( \text{OPT}(l, m) \) where \( m - l < j - i \).
Dynamic Programming Algorithm

\[ \text{OPT}(i, j) = \max\left( \text{OPT}(i, j - 1), \max_t \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right) \]

- There are \( O(n^2) \) sub-problems.
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- Note that computing \( \text{OPT}(i, j) \) involves sub-problems \( \text{OPT}(l, m) \) where \( m - l < j - i \).

---

Initialize \( \text{OPT}(i, j) = 0 \) whenever \( i \geq j - 4 \)

For \( k = 5, 6, \ldots, n - 1 \)
  For \( i = 1, 2, \ldots, n - k \)
    Set \( j = i + k \)
    Compute \( \text{OPT}(i, j) \) using the recurrence in (6.13)
  Endfor
Endfor

Return \( \text{OPT}(1, n) \)
Dynamic Programming Algorithm

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_{t} (1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1)) \right)
\]

- There are \(O(n^2)\) sub-problems.
- How do we order them from “smallest” to “largest”?
- Note that computing \(\text{OPT}(i, j)\) involves sub-problems \(\text{OPT}(l, m)\) where \(m - l < j - i\).

\[
\text{Initialize } \text{OPT}(i, j) = 0 \text{ whenever } i \geq j - 4 \\
\text{For } k = 5, 6, \ldots, n - 1 \\
\hspace{1em} \text{For } i = 1, 2, \ldots, n - k \\
\hspace{2em} \text{Set } j = i + k \\
\hspace{2em} \text{Compute } \text{OPT}(i, j) \text{ using the recurrence in (6.13)} \\
\hspace{1em} \text{Endfor} \\
\text{Endfor} \\
\text{Return } \text{OPT}(1, n)
\]

- Running time of the algorithm is \(O(n^3)\).
**Example of Algorithm**

RNA sequence *ACCGCUAGU*

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Filling in the values for $k = 5$

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Filling in the values for $k = 8$
Motivation

▶ Computational finance:
  ▶ Each node is a financial agent.
  ▶ The cost $c_{uv}$ of an edge $(u, v)$ is the cost of a transaction in which we buy from agent $u$ and sell to agent $v$.
  ▶ Negative cost corresponds to a profit.

▶ Internet routing protocols
  ▶ Dijkstra’s algorithm needs knowledge of the entire network.
  ▶ Routers only know which other routers they are connected to.
  ▶ Algorithm for shortest paths with negative edges is decentralised.
  ▶ We will not study this algorithm in the class. See Chapter 6.9.
Problem Statement

▶ Input: a directed graph \( G = (V, E) \) with a cost function \( c : E \to \mathbb{R} \), i.e., \( c_{uv} \) is the cost of the edge \((u, v) \in E\).

▶ A negative cycle is a directed cycle whose edges have a total cost that is negative.

▶ Two related problems:
  1. If \( G \) has no negative cycles, find the shortest s-t path: a path of from source \( s \) to destination \( t \) with minimum total cost.
  2. Does \( G \) have a negative cycle?
Problem Statement

Input: a directed graph $G = (V, E)$ with a cost function $c : E \rightarrow \mathbb{R}$, i.e., $c_{uv}$ is the cost of the edge $(u, v) \in E$.

A negative cycle is a directed cycle whose edges have a total cost that is negative.

Two related problems:

1. If $G$ has no negative cycles, find the shortest s-t path: a path of from source $s$ to destination $t$ with minimum total cost.
2. Does $G$ have a negative cycle?

Figure 6.20 In this graph, one can find s-t paths of arbitrarily negative cost (by going around the cycle $C$ many times).
Approaches for Shortest Path Algorithm

1. Dijsktra’s algorithm.

2. Add some large constant to each edge.
**Approaches for Shortest Path Algorithm**

1. **Dijkstra’s algorithm.** Computes incorrect answers because it is greedy.

2. **Add some large constant to each edge.** Computes incorrect answers because the minimum cost path changes.

![Diagram](attachment:attachment.png)

**Figure 6.21** (a) With negative edge costs, Dijkstra’s Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest s-t path.
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is *simple* (does not repeat a node)
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is \textit{simple} (does not repeat a node) and hence has at most $n - 1$ edges.
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is *simple* (does not repeat a node) and hence has at most $n - 1$ edges.
- How do we define sub-problems?
Assume $G$ has no negative cycles.

Claim: There is a shortest path from $s$ to $t$ that is simple (does not repeat a node) and hence has at most $n - 1$ edges.

How do we define sub-problems?

- Shortest $s$-$t$ path has $\leq n - 1$ edges: how we can reach $t$ using $i$ edges, for different values of $i$?
- We do not know which nodes will be in shortest $s$-$t$ path: how we can reach $t$ from each node in $V$?
Assume $G$ has no negative cycles.
Claim: There is a shortest path from $s$ to $t$ that is simple (does not repeat a node) and hence has at most $n - 1$ edges.

How do we define sub-problems?

- Shortest $s$-$t$ path has $\leq n - 1$ edges: how we can reach $t$ using $i$ edges, for different values of $i$?
- We do not know which nodes will be in shortest $s$-$t$ path: how we can reach $t$ from each node in $V$?

Sub-problems defined by varying the number of edges in the shortest path and by varying the starting node in the shortest path.
Dynamic Programming Recursion

- $\text{OPT}(i, v)$: minimum cost of a $v$-$t$ path that uses at most $i$ edges.
- $t$ is not explicitly mentioned in the sub-problems.
- Goal is to compute $\text{OPT}(n - 1, s)$. 

1. If $P$ actually uses $i - 1$ edges, then $\text{OPT}(i, v) = \text{OPT}(i - 1, v)$.
2. If first node on $P$ is $w$, then $\text{OPT}(i, v) = c_{vw} + \text{OPT}(i - 1, w)$.

$\text{OPT}(i, v) = \min(\text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)))$
**Dynamic Programming Recursion**

- **OPT(i, v)**: minimum cost of a v-t path that uses at most i edges.
- t is not explicitly mentioned in the sub-problems.
- Goal is to compute OPT(n − 1, s).

![Diagram](image.png)

**Figure 6.22** The minimum-cost path $P$ from $v$ to $t$ using at most $i$ edges.

- Let $P$ be the optimal path whose cost is OPT(i, v).

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T. M. Murali March 17, 22, 24, 2016 Dynamic Programming
**Dynamic Programming Recursion**

- \( \mathsf{OPT}(i, v) \): minimum cost of a \( v \)-\( t \) path that uses at most \( i \) edges.
- \( t \) is not explicitly mentioned in the sub-problems.
- Goal is to compute \( \mathsf{OPT}(n - 1, s) \).

\[
\mathsf{OPT}(i, v) = \min\left( \mathsf{OPT}(i - 1, v), c_{vw} + \mathsf{OPT}(i - 1, w) \right)
\]

**Figure 6.22** The minimum-cost path \( P \) from \( v \) to \( t \) using at most \( i \) edges.

- Let \( P \) be the optimal path whose cost is \( \mathsf{OPT}(i, v) \).
  1. If \( P \) actually uses \( i - 1 \) edges, then \( \mathsf{OPT}(i, v) = \mathsf{OPT}(i - 1, v) \).
  2. If first node on \( P \) is \( w \), then \( \mathsf{OPT}(i, v) = c_{vw} + \mathsf{OPT}(i - 1, w) \).
Dynamic Programming Recursion

- \( OPT(i, v) \): minimum cost of a \( v \)-\( t \) path that uses at most \( i \) edges.
- \( t \) is not explicitly mentioned in the sub-problems.
- Goal is to compute \( OPT(n - 1, s) \).

Let \( P \) be the optimal path whose cost is \( OPT(i, v) \).

1. If \( P \) actually uses \( i - 1 \) edges, then \( OPT(i, v) = OPT(i - 1, v) \).
2. If first node on \( P \) is \( w \), then \( OPT(i, v) = c_{vw} + OPT(i - 1, w) \).

\[
OPT(i, v) = \min \left( OPT(i - 1, v), \min_{w \in V} (c_{vw} + OPT(i - 1, w)) \right)
\]
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, \nu) = \min \left( \text{OPT}(i - 1, \nu), \min_{w \in V} \left( c_{vw} + \text{OPT}(i - 1, w) \right) \right) \]
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]
Example of Dynamic Programming Recursion

$$\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right)$$
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} \left( c_{vw} + \text{OPT}(i - 1, w) \right) \right) \]
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]
Example of Dynamic Programming Recursion

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\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right)
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\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} \left( c_{vw} + \text{OPT}(i - 1, w) \right) \right) \]
Example of Dynamic Programming Recursion

$$\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} \left( c_{vw} + \text{OPT}(i - 1, w) \right) \right)$$

![Graph example with dynamic programming table]

- Table entries represent the minimum cost to reach node $v$ from node $0$.
- Red highlighted entries indicate the optimal solution path.

T. M. Murali  March 17, 22, 24, 2016 Dynamic Programming
Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]
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Example of Dynamic Programming Recursion

$$\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right)$$

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 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{t} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{a} & \infty & -3 & -3 & -4 & -6 & -6 \\
\text{b} & \infty & \infty & 0 & -2 & -2 & -2 \\
\text{c} & 8 & 3 & 3 & 3 & 3 & 3 \\
\text{d} & 8 & 4 & 3 & 3 & 3 & 0 \\
\text{e} & \infty & 2 & 0 & 0 & 0 & 0 \\
\end{array} \]
Alternate Dynamic Programming Formulation

- $OPT_{i,v}$: minimum cost of a $v$-$t$ path that uses exactly $i$ edges. Goal is to compute
Alternate Dynamic Programming Formulation

- \( OPT(i, v) \): minimum cost of a \( v-t \) path that uses exactly \( i \) edges. Goal is to compute

\[
\min_{i=1}^{n-1} OPT(i, s).
\]
Alternate Dynamic Programming Formulation

- \( OPT_{\leq}(i, v) \): minimum cost of a \( v-t \) path that uses exactly \( i \) edges. Goal is to compute

\[
\min_{i=1}^{n-1} \text{OPT}_{\leq}(i, s).
\]

- Let \( P \) be the optimal path whose cost is \( \text{OPT}_{\leq}(i, v) \).
Alternate Dynamic Programming Formulation

- $OPT_{v-t}(i, v)$: minimum cost of a $v-t$ path that uses exactly $i$ edges. Goal is to compute

$$\min_{i=1}^{n-1} OPT_{v-t}(i, s).$$

- Let $P$ be the optimal path whose cost is $OPT_{v-t}(i, v)$.
  - If first node on $P$ is $w$, then $OPT_{v-t}(i, v) = c_{vw} + OPT_{v-t}(i - 1, w)$. 
Alternate Dynamic Programming Formulation

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  \[
  OPT_{\leq}(i, v) = \min_{w \in V} (c_{vw} + OPT_{\leq}(i - 1, w))
  \]
Alternate Dynamic Programming Formulation

- $OPT_{=} (i, v)$: minimum cost of a $v$-$t$ path that uses exactly $i$ edges. Goal is to compute

$$
\min_{i=1}^{n-1} \text{OPT}_{=} (i, s).
$$

- Let $P$ be the optimal path whose cost is $OPT_{=} (i, v)$.
  - If first node on $P$ is $w$, then $OPT_{=} (i, v) = c_{vw} + OPT_{=} (i - 1, w)$.

$$
OPT_{=} (i, v) = \min_{w \in V} \left( c_{vw} + OPT_{=} (i - 1, w) \right)
$$

- Compare the two desired solutions:

$$
\min_{i=1}^{n-1} \text{OPT}_{=} (i, s) = \min_{i=1}^{n-1} \left( \min_{w \in V} \left( c_{sw} + OPT_{=} (i - 1, w) \right) \right)
$$

$$
OPT (n - 1, s) = \min \left( OPT (n - 2, s), \min_{w \in V} \left( c_{sw} + OPT (n - 2, w) \right) \right)
$$
Bellman-Ford Algorithm

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]

```
Shortest-Path(G, s, t)
    n = number of nodes in G
    Array M[0,..n - 1, V]
    Define M[0, t] = 0 and M[0, v] = \infty for all other v \in V
    For i = 1, ..., n - 1
        For v \in V in any order
            Compute M[i, v] using the recurrence (6.23)
        Endfor
    Endfor
    Return M[n - 1, s]
```
Bellman-Ford Algorithm

\[
\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right)
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Shortest-Path(G, s, t)

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  - For \(v \in V\) in any order
    - Compute \(M[i, v]\) using the recurrence (6.23)
  - Endfor
- Endfor
- Return \(M[n-1, s]\)

- Space used is \(O(n^2)\). Running time is \(O(n^3)\).
- If shortest path uses \(k\) edges, we can recover it in \(O(kn)\) time by tracing back through smaller sub-problems.
An Improved Bound on the Running Time

- Suppose $G$ has $n$ nodes and $m \ll \binom{n}{2}$ edges. Can we demonstrate a better upper bound on the running time?
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Suppose $G$ has $n$ nodes and $m \ll \binom{n}{2}$ edges. Can we demonstrate a better upper bound on the running time?

$$M[i, v] = \min (M[i - 1, v], \min_{w \in N_v} (c_{vw} + M[i - 1, w]))$$

- $w$ only needs to range over outgoing neighbours $N_v$ of $v$.
- If $n_v = |N_v|$ is the number of outgoing neighbours of $v$, then in each round, we spend time equal to

$$\sum_{v \in V} n_v = \ldots$$
An Improved Bound on the Running Time

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$$\sum_{v \in V} n_v = m.$$

- The total running time is $O(mn)$.  

T. M. Murali March 17, 22, 24, 2016
Dynamic Programming
Improving the Memory Requirements

\[ M[i, v] = \min \left( M[i - 1, v], \min_{w \in N_v} (c_{vw} + M[i - 1, w]) \right) \]

- The algorithm uses \( O(n^2) \) space to store the array \( M \).
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- Observe that \( M[i, v] \) depends only on \( M[i - 1, \ast] \) and no other indices.
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- Modified algorithm:
  1. Maintain two arrays \( M \) and \( M' \) indexed over \( V \).
  2. At the beginning of each iteration, copy \( M \) into \( M' \).
  3. To update \( M \), use

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]
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  1. Maintain two arrays \( M \) and \( M' \) indexed over \( V \).
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  3. To update \( M \), use

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

- Claim: at the beginning of iteration \( i \), \( M \) stores values of \( \text{OPT}(i - 1, v) \) for all nodes \( v \in V \).
- Space used is \( O(n) \).
Computing the Shortest Path: Algorithm

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

- How can we recover the shortest path that has cost \( M[v] \)?

- For each node \( v \), compute and update \( f(v) \), the first node after \( v \) in the current shortest path from \( v \) to \( t \).

- Updating \( f(v) \):
  - If \( x \) is the node that attains the minimum in \( \min_{w \in N_v} (c_{vw} + M'[w]) \), set \( M[v] = c_{vx} + M'[x] \) and \( f(v) = x \).

- At the end, follow \( f(v) \) pointers from \( s \) to \( t \).
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Example of Maintaining Pointers

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]
\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]
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\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]
Computing the Shortest Path: Correctness

- **Pointer graph** $P(V, F)$: each edge in $F$ is $(v, f(v))$.
  - Can $P$ have cycles?
  - Is there a path from $s$ to $t$ in $P$?
  - Can there be multiple paths $s$ to $t$ in $P$?
  - Which of these is the shortest path?

![Diagram of a pointer graph with edge weights](image)

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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v], \min_{w \in \mathcal{N}_v} (c_{vw} + M'[w]) \right)$

- Claim: If $P$ has a cycle $C$, then $C$ has negative cost.
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- **Claim**: If $P$ has a cycle $C$, then $C$ has negative cost.
  - Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M[w]$.
  - Between this assignment and the assignment of $f(v)$ to some other node, $M[w]$ may itself decrease. Hence, $M[v] \geq c_{vw} + M[w]$, in general.

---

$V_1 \rightarrow V_2 \rightarrow V_3$

$V_5 \leftarrow V_4$
Computing the Shortest Path: Cycles in $P$

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  - Let $v_1, v_2, \ldots v_k$ be the nodes in $C$ and assume that $(v_k, v_1)$ is the last edge to have been added.
  - What is the situation just before this addition?
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v] , \min_{w \in N_v} (c_{vw} + M'[w]) \right)$

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  - What is the situation just before this addition?
  - $M[v_i] - M[v_{i+1}] \geq c_{v_iv_{i+1}}$, for all $1 \leq i < k - 1$.
  - $M[v_k] - M[v_1] > c_{v_kv_1}$.

$\rightarrow$
Computing the Shortest Path: Cycles in $P$

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

▶ Claim: If $P$ has a cycle $C$, then $C$ has negative cost.
▶ Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M[w]$.
▶ Between this assignment and the assignment of $f(v)$ to some other node, $M[w]$ may itself decrease. Hence, $M[v] \geq c_{vw} + M[w]$, in general.
▶ Let $v_1, v_2, \ldots, v_k$ be the nodes in $C$ and assume that $(v_k, v_1)$ is the last edge to have been added.
▶ What is the situation just before this addition?
▶ $M[v_i] - M[v_{i+1}] \geq c_{v_i,v_{i+1}}$, for all $1 \leq i < k - 1$.
▶ $M[v_k] - M[v_1] > c_{v_k,v_1}$.
▶ Adding all these inequalities, $0 > \sum_{i=1}^{k-1} c_{v_i,v_{i+1}} + c_{v_k,v_1} = \text{cost of } C$. 

Corollary: if $G$ has no negative cycles that $P$ does not either.
Computing the Shortest Path: Cycles in \( P \)

\[
M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)
\]

▶ Claim: If \( P \) has a cycle \( C \), then \( C \) has negative cost.

▶ Suppose we set \( f(v) = w \). At this instant, \( M[v] = c_{vw} + M[w] \).

▶ Between this assignment and the assignment of \( f(v) \) to some other node, \( M[w] \) may itself decrease. Hence, \( M[v] \geq c_{vw} + M[w] \), in general.

▶ Let \( v_1, v_2, \ldots, v_k \) be the nodes in \( C \) and assume that \((v_k, v_1)\) is the last edge to have been added.

▶ What is the situation just before this addition?

▶ \( M[v_i] - M[v_{i+1}] \geq c_{v_i v_{i+1}} \), for all \( 1 \leq i < k - 1 \).

▶ \( M[v_k] - M[v_1] > c_{v_k v_1} \).

▶ Adding all these inequalities, \( 0 > \sum_{i=1}^{k-1} c_{v_i v_{i+1}} + c_{v_k v_1} = \text{cost of } C \).

▶ Corollary: if \( G \) has no negative cycles that \( P \) does not either.
Computing the Shortest Path: Paths in $P$

- Let $P$ be the pointer graph upon termination of the algorithm.
- Consider the path $P_v$ in $P$ obtained by following the pointers from $v$ to $f(v) = v_1$, to $f(v_1) = v_2$, and so on.

Claim: $P_v$ terminates at $t$.

Claim: $P_v$ is the shortest path in $G$ from $v$ to $t$. 

T. M. Murali March 17, 22, 24, 2016 Dynamic Programming
Computing the Shortest Path: Paths in $P$

- Let $P$ be the pointer graph upon termination of the algorithm.
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Computing the Shortest Path: Paths in $P$

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- Claim: $P_v$ terminates at $t$.
- Claim: $P_v$ is the shortest path in $G$ from $v$ to $t$. 
Bellman-Ford Algorithm: One Array

\[ M[v] = \min \left( M[v], \min_{w \in N_v} (c_{vw} + M[w]) \right) \]

- We can prove algorithm’s correctness in this case as well.
Bellman-Ford Algorithm: Early Termination

\[ M[v] = \min \left( M[v], \min_{w \in N_v} (c_{vw} + M[w]) \right) \]

- In general, after \( i \) iterations, the path whose length is \( M[v] \) may have many more than \( i \) edges.
Bellman-Ford Algorithm: Early Termination

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- In general, after \( i \) iterations, the path whose length is \( M[v] \) may have many more than \( i \) edges.
- Early termination: If \( M \) does not change after processing all the nodes, we have computed all the shortest paths to \( t \).