Divide and Conquer Algorithms

T. M. Murali

March 1, 2016
Divide and Conquer

- Break up a problem into several parts.
- Solve each part recursively.
- Solve base cases by brute force.
- Efficiently combine solutions for sub-problems into final solution.

Common use:
- Partition problem into two equal sub-problems of size $n/2$.
- Solve each part recursively.
- Combine the two solutions in $O(n)$ time.
- Resulting running time is $O(n \log n)$. 

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Mergesort

Sort

INSTANCE: Nonempty list $L = x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

▶ Mergesort is a divide-and-conquer algorithm for sorting.
   1. Partition $L$ into two lists $A$ and $B$ of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively.
   2. Recursively sort $A$.
   3. Recursively sort $B$.
   4. Merge the sorted lists $A$ and $B$ into a single sorted list.
Merging Two Sorted Lists

- Merge two sorted lists \( A = a_1, a_2, \ldots, a_k \) and \( B = b_1, b_2, \ldots, b_l \).
  
  Maintain a *current* pointer for each list.
  
  Initialise each pointer to the front of the list.
  
  While both lists are nonempty:
    
    Let \( a_i \) and \( b_j \) be the elements pointed to by the *current* pointers.
    
    Append the smaller of the two to the output list.
    
    Advance the current pointer in the list that the smaller element belonged to.
  
  EndWhile
  
  Append the rest of the non-empty list to the output.
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  EndWhile
  Append the rest of the non-empty list to the output.

- Running time of this algorithm is $O(k + l)$. 
Analysing Mergesort

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Worst-case running time for \( n \) elements \( \leq \)
- Worst-case running time for \( \lfloor n/2 \rfloor \) elements +
- Worst-case running time for \( \lceil n/2 \rceil \) elements +
- Time to split the input into two lists +
- Time to merge two sorted lists.

Assume \( n \) is a power of 2.

Define \( T(n) \equiv \) Worst-case running time for \( n \) elements, for every \( n \geq 1 \).

\( T(n) \leq 2T(n/2) + cn \), \( n > 2 \)

Three basic ways of solving this recurrence relation:
1. "Unroll" the recurrence (somewhat informal method).
2. Guess a solution and substitute into recurrence to check.
3. Guess solution in \( O() \) form and substitute into recurrence to determine the constants.
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Unrolling the recurrence

- Recursion tree has \( \log n \) levels.
- Total work done at each level is \( cn \).
- Running time of the algorithm is \( cn \log n \).

Use this method only to get an idea of the solution.

Figure 5.1 Unrolling the recurrence \( T(n) \leq 2T(n/2) + O(n) \).
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Figure 5.1  Unrolling the recurrence \( T(n) \leq 2T(n/2) + O(n) \).
Substituting a Solution into the Recurrence

- Guess that the solution is $T(n) \leq cn \log n$ (logarithm to the base 2).
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- (Strong) Inductive hypothesis: assume $T(m) \leq cm \log_2 m$ for all $m < n$. 
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Why is $T(n) \leq kn^2$ a "loose" bound?

Why doesn’t an attempt to prove $T(n) \leq kn$ for some $k > 0$ work?
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T(n) \leq 2T \left( \frac{n}{2} \right) + cn
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\leq 2 \left( \frac{cn}{2} \log \left( \frac{n}{2} \right) \right) + cn, \text{ by the inductive hypothesis}
\]

\[
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- Why doesn’t an attempt to prove $T(n) \leq kn$, for some $k > 0$ work?
Partial Substitution

- Guess that the solution is $kn \log n$ (logarithm to the base 2).
- Substitute guess into the recurrence relation to check what value of $k$ will satisfy the recurrence relation.
Partial Substitution

- Guess that the solution is $kn \log n$ (logarithm to the base 2).
- Substitute guess into the recurrence relation to check what value of $k$ will satisfy the recurrence relation.
- $k \geq c$ will work.
Proof for All Values of $n$

- We assumed $n$ is a power of 2.
- How do we generalise the proof?
Proof for All Values of $n$

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- How do we generalise the proof?
- Basic axiom: $T(n) \leq T(n + 1)$, for all $n$: worst case running time increases as input size increases.
- Let $m$ be the smallest power of 2 larger than $n$.
- $T(n) \leq T(m) = O(m \log m)$
Proof for All Values of $n$

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- Basic axiom: $T(n) \leq T(n + 1)$, for all $n$: worst case running time increases as input size increases.
- Let $m$ be the smallest power of 2 larger than $n$.
- $T(n) \leq T(m) = O(m \log m) = O(n \log n)$, because $m \leq 2n$. 
Other Recurrence Relations

- Divide into $q$ sub-problems of size $n/2$ and merge in $O(n)$ time. Two distinct cases: $q = 1$ and $q > 2$.
- Divide into two sub-problems of size $n/2$ and merge in $O(n^2)$ time.
$T(n) = qT(n/2) + cn$, $q = 1$

$cn$ time, plus recursive calls

Level 0: $cn$ total

Level 1: $cn/2$ total

Level 2: $cn/4$ total

**Figure 5.3** Unrolling the recurrence $T(n) \leq T(n/2) + O(n)$. 
\[ T(n) = qT(n/2) + cn, \quad q = 1 \]

Each invocation reduces the problem size by a factor of 2 \( \Rightarrow \) there are \( \log n \) levels in the recursion tree.

At level \( i \) of the tree, the problem size is \( n/2^i \) and the work done is \( cn/2^i \).

Therefore, the total work done is

\[
\sum_{i=0}^{i=\log n} \frac{cn}{2^i} = O(n).
\]

**Figure 5.3** Unrolling the recurrence \( T(n) \leq T(n/2) + O(n) \).
\[ T(n) = qT(n/2) + cn, \quad q > 2 \]

There are \( \log n \) levels in the recursion tree.

At level \( i \) of the tree, there are \( q^i \) sub-problems, each of size \( n/2^i \).

The total work done at level \( i \) is \( q^i cn/2^i \). Therefore, the total work done is

\[
T(n) \leq \sum_{i=0}^{\log 2 n} q^i cn/2^i 
\]

\[
= cn \sum_{i=0}^{\log 2 n} (q^2/4)^i 
\]

\[
= \frac{cn}{1 - q^2/4} 
\]

\[
= O\left(cn \left(\frac{q^2}{4}\right)^{\log 2 n}\right) 
\]

\[
= O\left(cn n \log \frac{q}{2 n}\right) 
\]

\[
= O\left(n \log \frac{q}{2 n}\right) 
\]

**Figure 5.2** Unrolling the recurrence \( T(n) \leq 3T(n/2) + O(n) \).
\[ T(n) = qT(n/2) + cn, \quad q > 2 \]

- There are \( \log n \) levels in the recursion tree.
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\[ T(n) \leq \sum_{i=0}^{i=\log_2 n} q^i \frac{cn}{2^i} \leq \]

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\[
T(n) \leq \sum_{i=0}^{i=\log_2 n} q^i \frac{cn}{2^i} \leq cn \sum_{i=0}^{i=\log_2 n} \left( \frac{q}{2} \right)^i
\]

\[
= O\left( cn \left( \frac{q}{2} \right)^{\log_2 n} \right) = O\left( cn \left( \frac{q}{2} \right)^{\left( \log_{q/2} n \right) \left( \log_2 q/2 \right)} \right)
\]

\[
= O\left( cn n^{\log_2 q/2} \right) = O\left( n^{\log_2 q} \right).
\]
\[ T(n) = 2T(n/2) + cn^2 \]

Total work done is

\[ \sum_{i=0}^{i=\log n} 2^i \left( \frac{cn}{2^i} \right)^2 \leq \]

\[ \leq O(n^2) \]
\[ T(n) = 2T(n/2) + cn^2 \]

- Total work done is

\[
\sum_{i=0}^{i=\log n} 2^i \left( \frac{cn}{2^i} \right)^2 \leq O(n^2).
\]