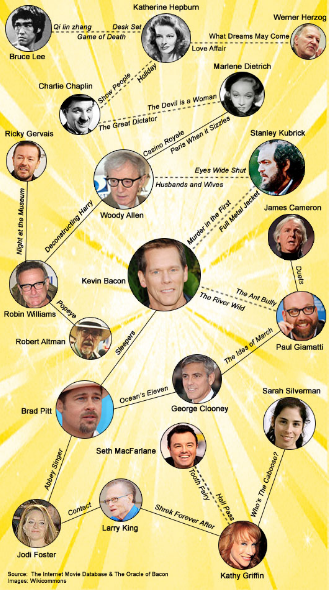
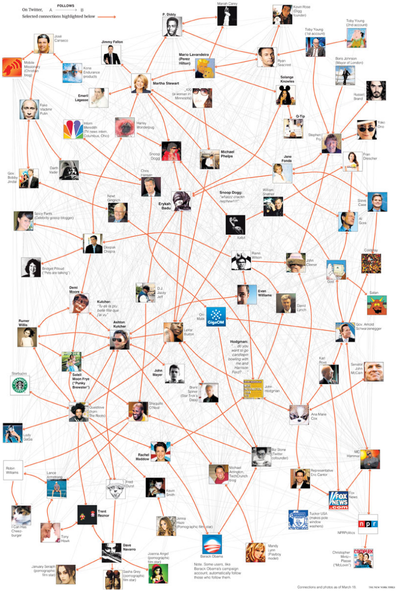
# Graphs

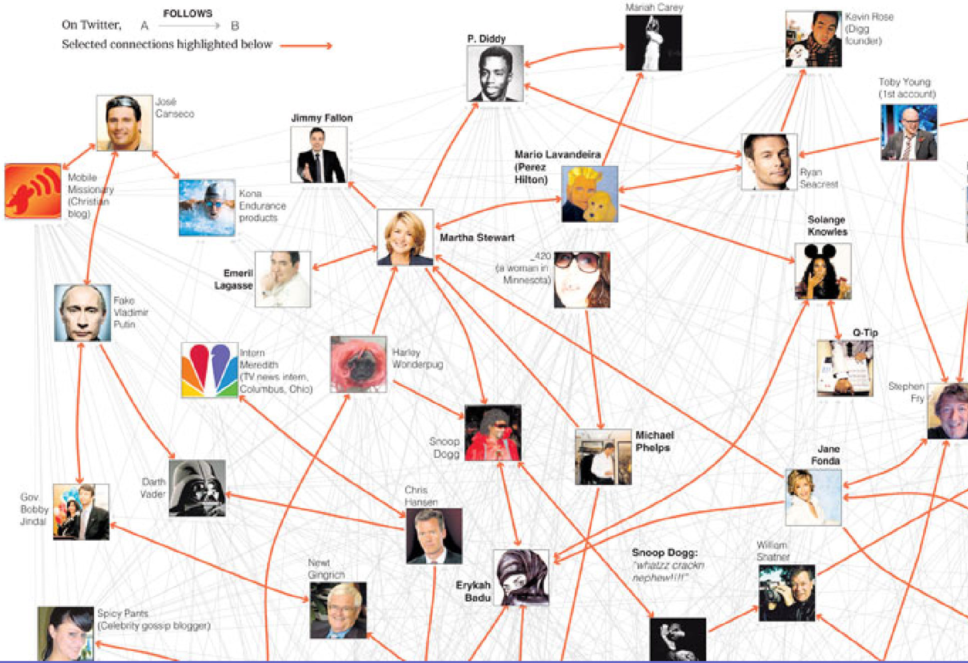
T. M. Murali

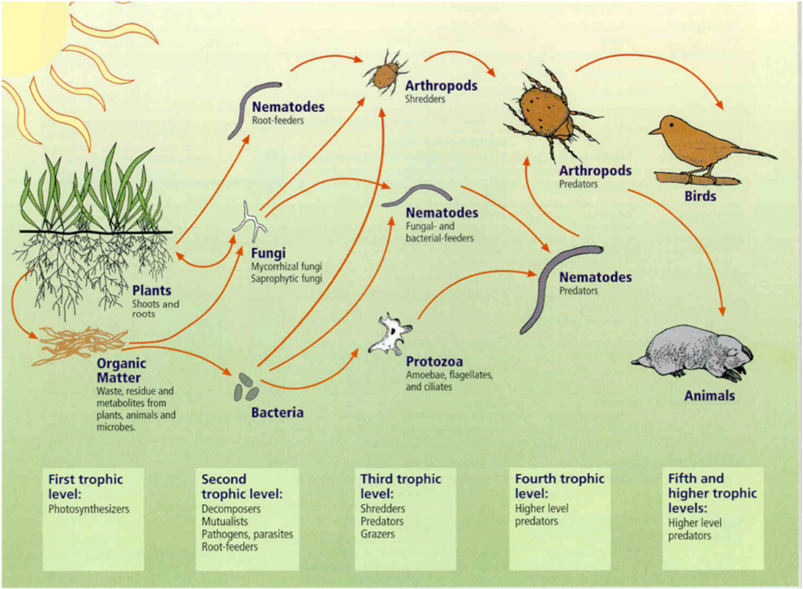
January 28, February 2, 4, 2016

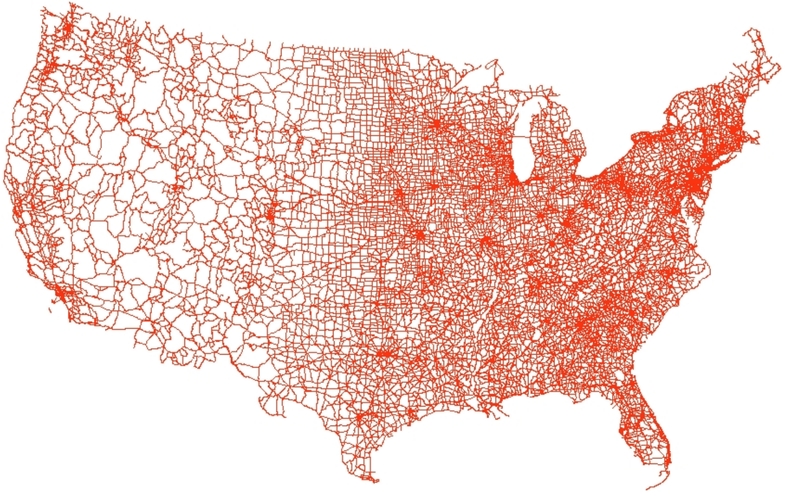


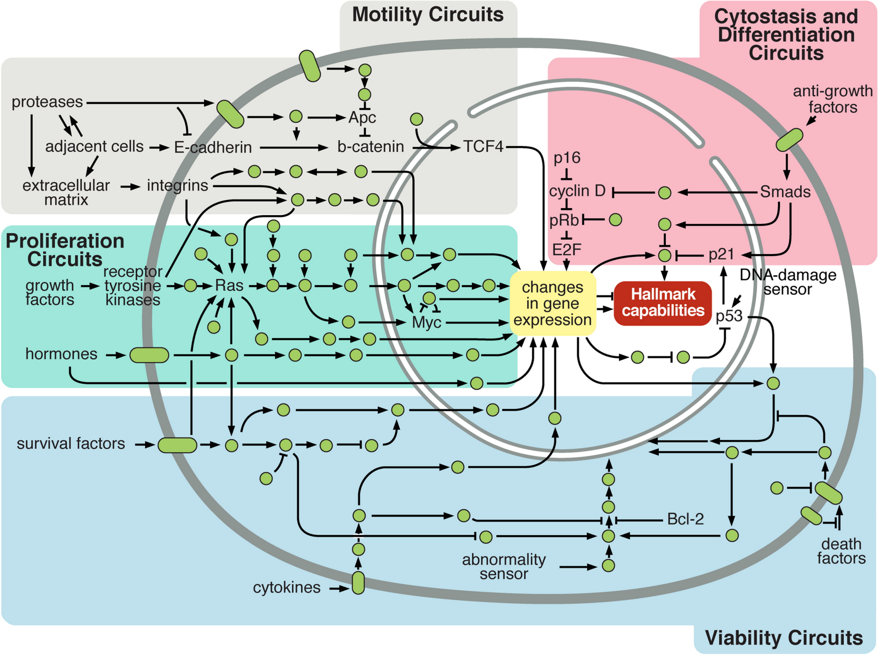

The Oracle of Bacon




On Twitter, A FOLLOWS B
Selected connections highlighted below
Mariah Carey
Kevin Rose (Digg founder)
P. Diddy
Toby Young (1st account)
José Canseco
Jimmy Fallon
Mario Lavandeira (Perez Hilton)
Ryan Seacrest
Mobile Missionary (Christian blog)
Kona Endurance products
Martha Stewart
Solange Knowles
_420 (a woman in Minnesota)
Emeril Lagasse
Fake Vladimir Putin
Q-Tip
Intern Meredith (TV news intern, Columbus, Ohio)
Harley Wonderpug
Stephen Fry
Snoop Dogg
Michael Phelps
Jane Fonda
Darth Vader
Gov. Bobby Jindal
Chris Hansen
Snoop Dogg: "whatzz crackn nephew!!!!"
William Shatner
Erykah Badu
Newt Gingrich
Spicy Pants (Celebrity gossip blogger)


Nematodes
Root-feeders
Arthropods
Shredders
Arthropods
Predators
Birds
Nematodes
Fungal- and
bacterial-feeders
Fungi
Mycorrhizal fungi
Saprophytic fungi
Plants
Shoots and
roots
Nematodes
Predators
Protozoa
Amoebae, flagellates,
and ciliates
Organic
Matter
Waste, residue and
metabolites from
plants, animals and
microbes.
Bacteria
Animals
First trophic
level:
Photosynthesizers
Second
trophic level:
Decomposers
Mutualists
Pathogens, parasites
Root-feeders
Third trophic
level:
Shredders
Predators
Grazers
Fourth trophic
level:
Higher level
predators
Fifth and
higher trophic
levels:
Higher level
predators




Motility Circuits
Cytostasis and Differentiation Circuits
Proliferation Circuits
Viability Circuits
proteases
adjacent cells
E-cadherin
extracellular matrix
integrins
Apc
b-catenin
TCF4
p16
cyclin D
pRb
E2F
anti-growth factors
Smads
p21
DNA-damage sensor
p53
growth factors
receptor tyrosine kinases
Ras
Myc
hormones
changes in gene expression
Hallmark capabilities
survival factors
Bcl-2
abnormality sensor
cytokines
death factors

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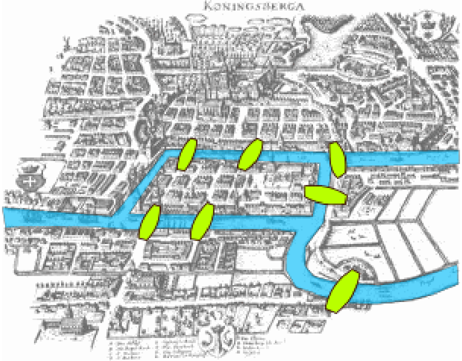

# Definition of a Graph

- *Undirected graph* $G = (V, E)$: set $V$ of nodes and set $E$ of edges, where $E \subseteq V \times V$. Elements of $E$ are unordered pairs.
  - Abuse of notation: write an edge $e$ between nodes $u$ and $v$ as $e = (u, v)$ and not as $e = \{u, v\}$.
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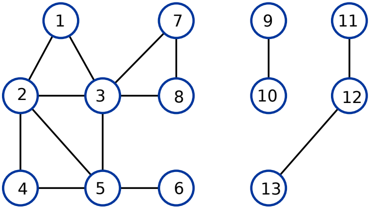
- By default, "graph" will mean an "undirected graph".

# Paths and Connectivity

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  - $P$ is called a path *from $v_1$ to $v_K$* or a *$v_1$-$v_k$* path.
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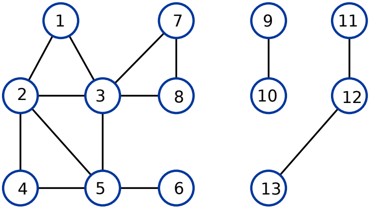


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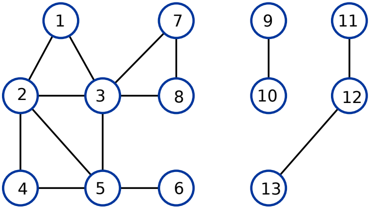


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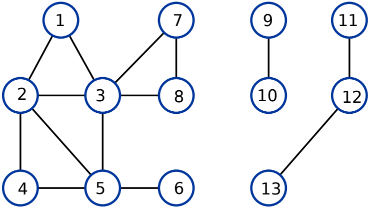


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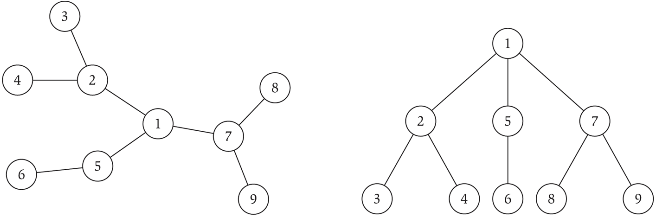
- *Distance* between two nodes $u$ and $v$ is the minimum number of edges in any $u$-$v$ path.

# Trees

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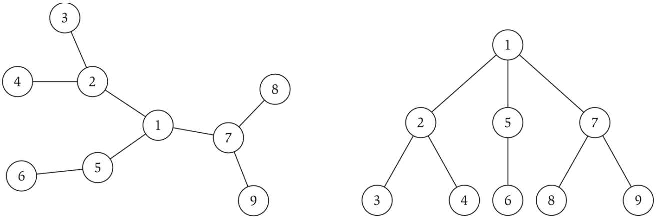


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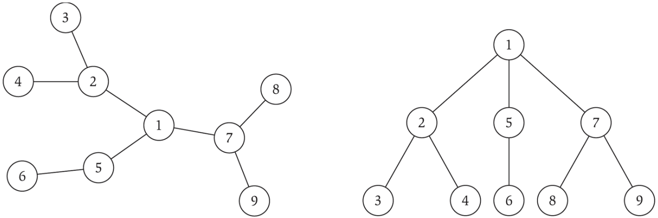


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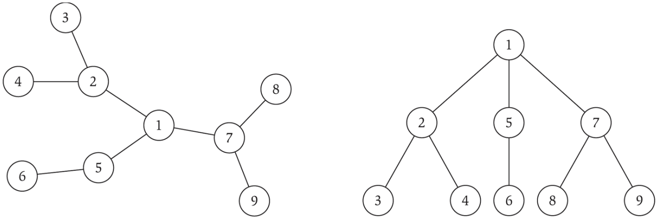


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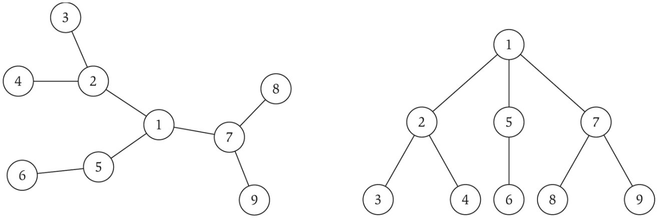


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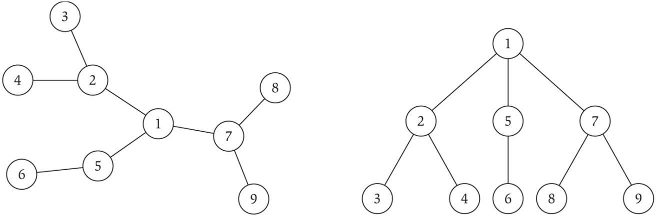


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- Examples of (rooted) trees: organisational hierarchy, class hierarchies in object-oriented languages.

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- Stronger claim: Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements implies the third:
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  - 2 and 3 $\Rightarrow$ 1:

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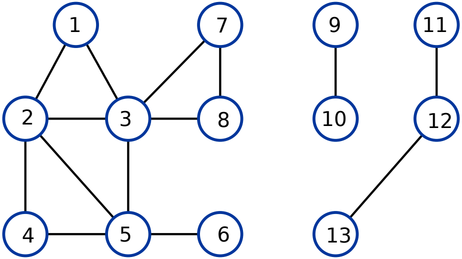
  - 2 and 3 $\Rightarrow$ 1: prove by contradiction.
  - 3 and 1 $\Rightarrow$ 2: prove yourself.

# $s$-$t$ Connectivity

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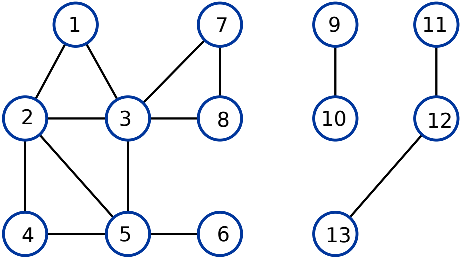


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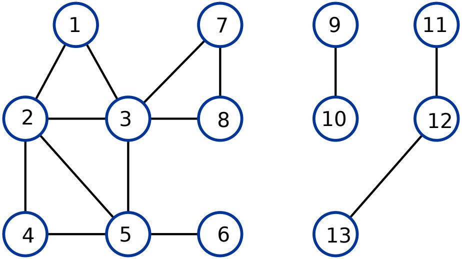


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- The *connected component of $G$ containing $s$* is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$.
- Algorithm for the $s$-$t$ Connectivity problem: compute the connected component of $G$ that contains $s$ and check if $t$ is in that component.

# Computing Connected Components

- "Explore" $G$ starting from $s$ and maintain set $R$ of visited nodes.

---

$R$ will consist of nodes to which $s$ has a path

Initially $R = \{s\}$

While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$

  Add $v$ to $R$

Endwhile

---

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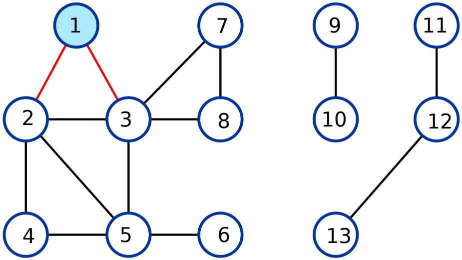

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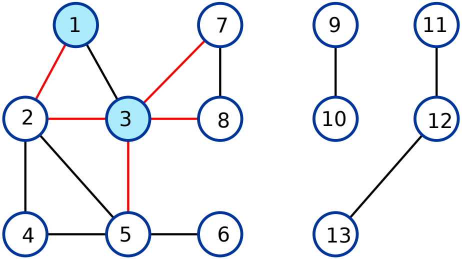

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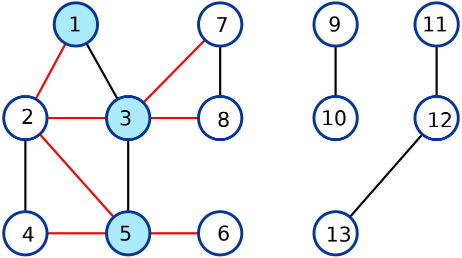

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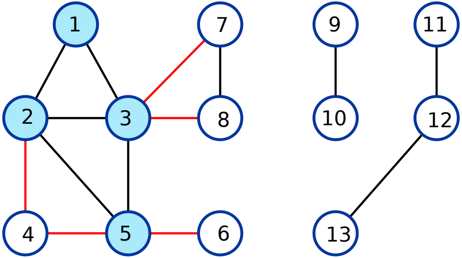

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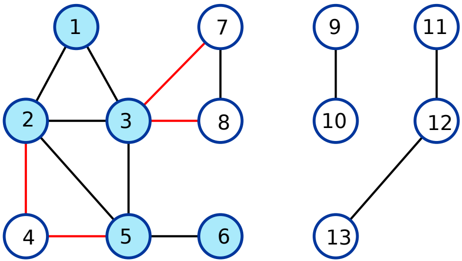

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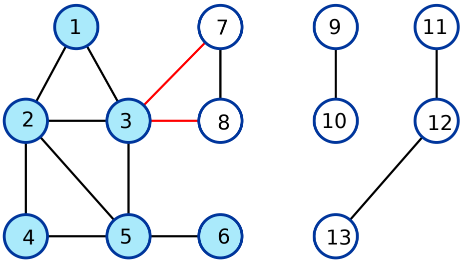

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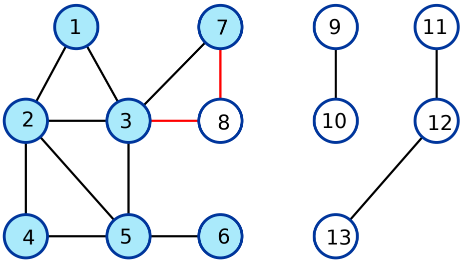

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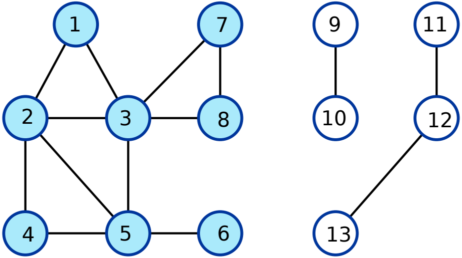

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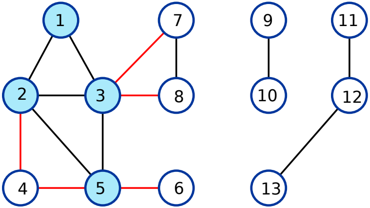


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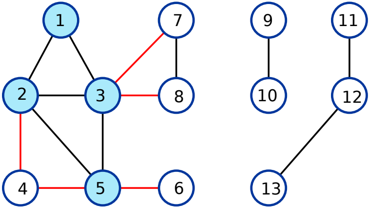


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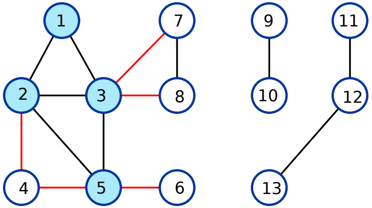


- How do we implement the `while` loop? Examine each edge in $E$.
- Other issues to consider:
  - Why does the algorithm terminate?
  - Does the algorithm truly compute connected component of $G$ containing $s$?
  - What is the running time of the algorithm?

# Termination of the Algorithm

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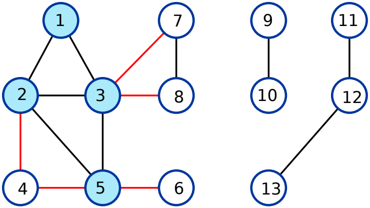


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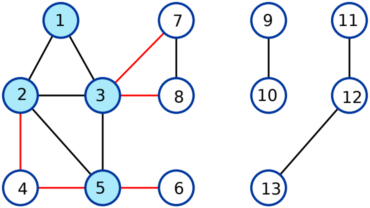


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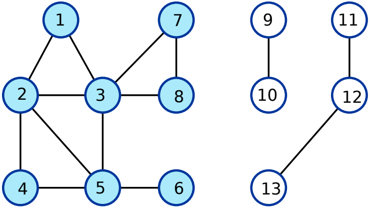


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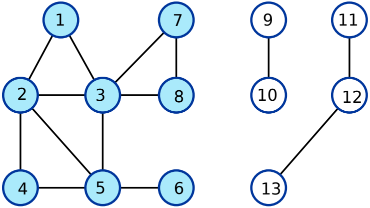


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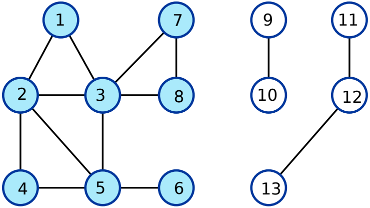


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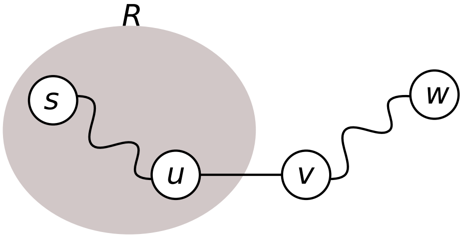
  - either $R = V$ at the end or
  - in the last iteration, every edge either has both nodes in $R$ or both nodes not in $R$.

# Correctness of the Algorithm

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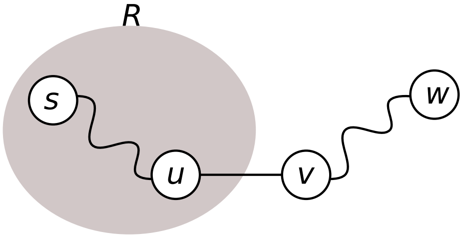


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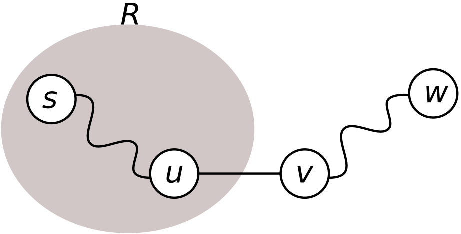


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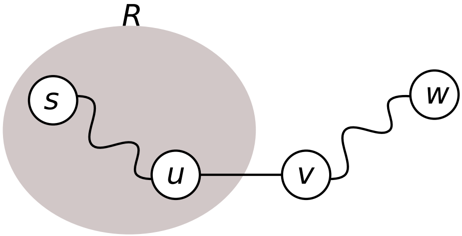


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  - Note: wrong to assume that predecessor of $w$ in $P$ is not in $R$.

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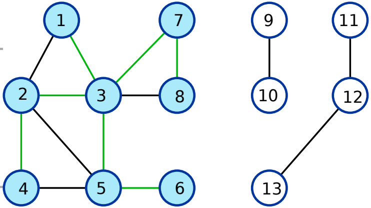


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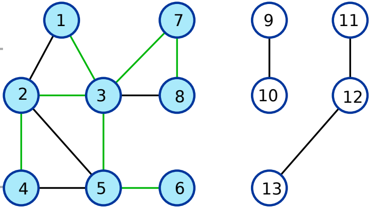


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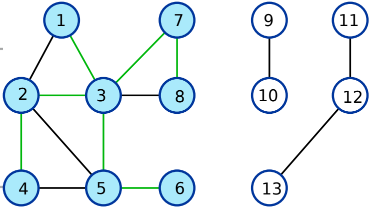


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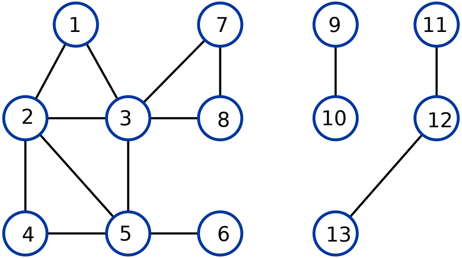
- Can we improve the running time by processing edges more carefully?

# Breadth-First Search (BFS)




- Idea: explore $G$ starting at $s$ and going "outward" in all directions, adding nodes one layer at a time.

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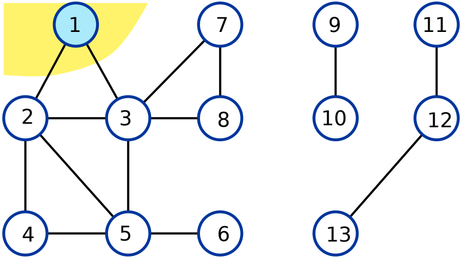


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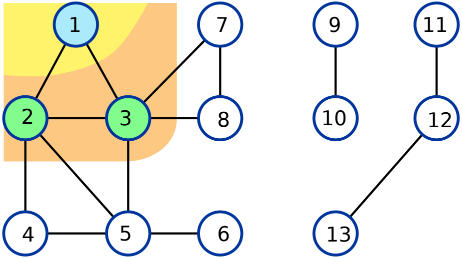


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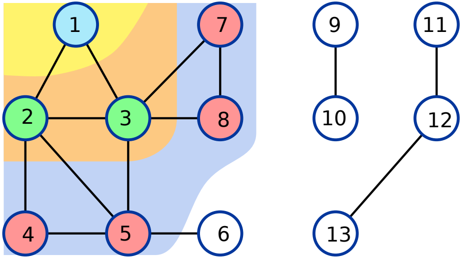


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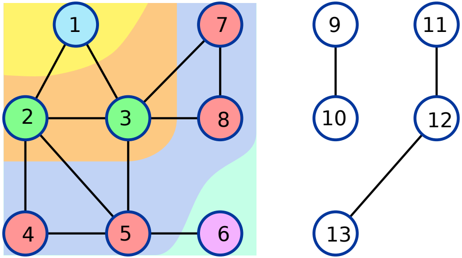


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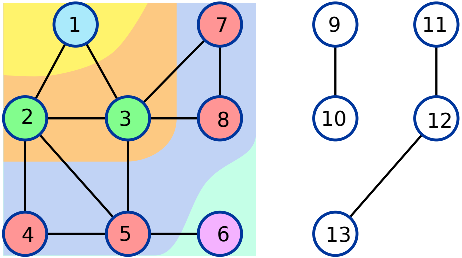
# Properties of BFS

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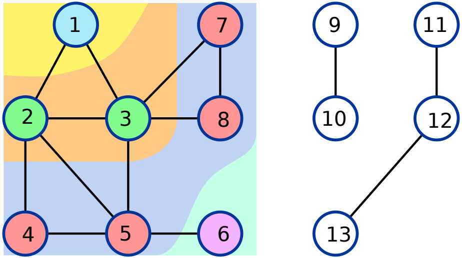


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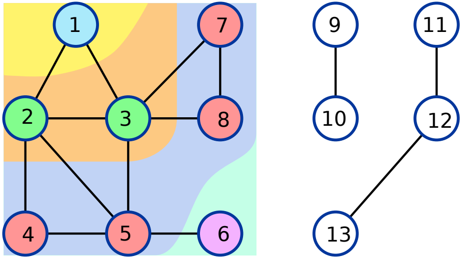


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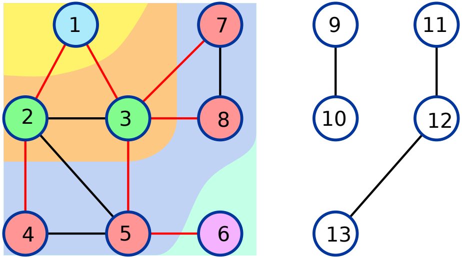


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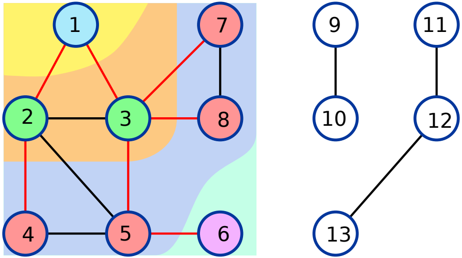


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  - Why is $T$ a tree?

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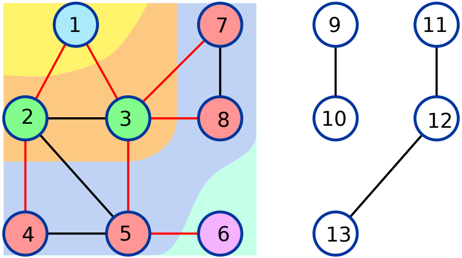


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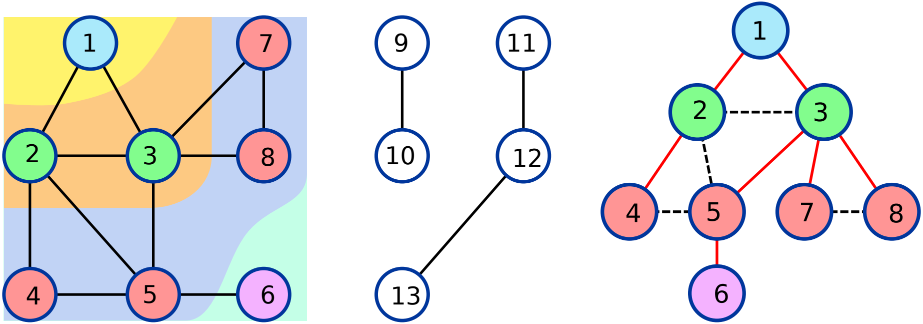
- Claim: There is a path from $s$ to $t$ if and only if $t$ is a member of some layer.
- Let $v$ be a node in layer $L_{j+1}$ and $u$ be the "first" node in $L_j$ such that $(u, v)$ is an edge in $G$. Consider the graph $T$ formed by all such edges, directed from $u$ to $v$.
  - Why is $T$ a tree? It is connected. The number of edges in $T$ is the number of nodes in all the layers minus 1.
  - $T$ is called the *breadth-first search tree*.

# BFS Trees

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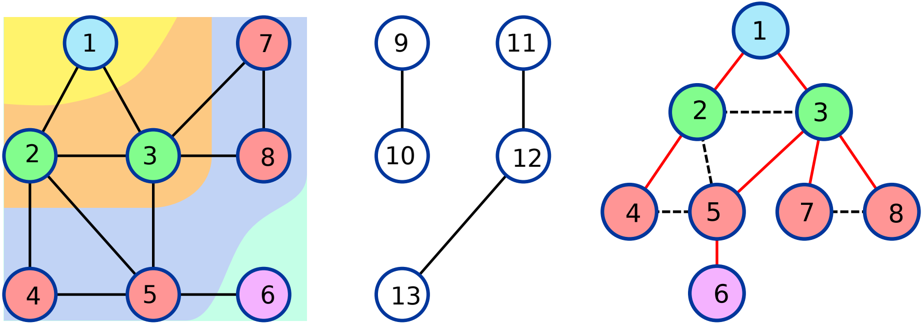


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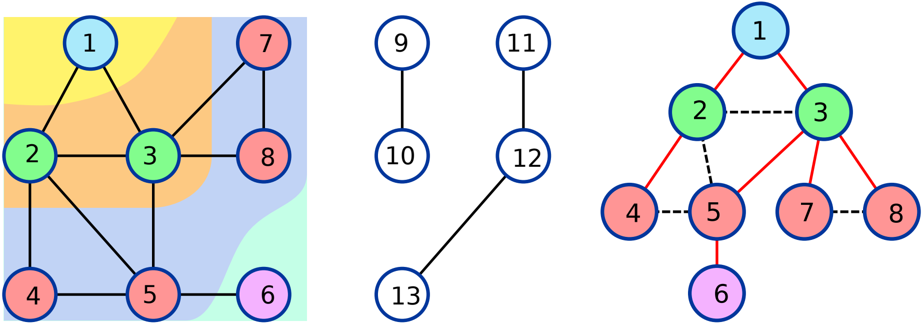


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- Still unresolved: an efficient implementation of BFS.

# Depth-First Search (DFS)

- Explore $G$ as if it were a maze: start from $s$, traverse first edge out (to node $v$), traverse first edge out of $v$, . . . , reach a dead-end, backtrack, . . . . . .

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- *Depth-first search tree* is a tree $T$: when DFS($v$) is invoked directly during the call to DFS($v$), add edge $(u, v)$ to $T$.

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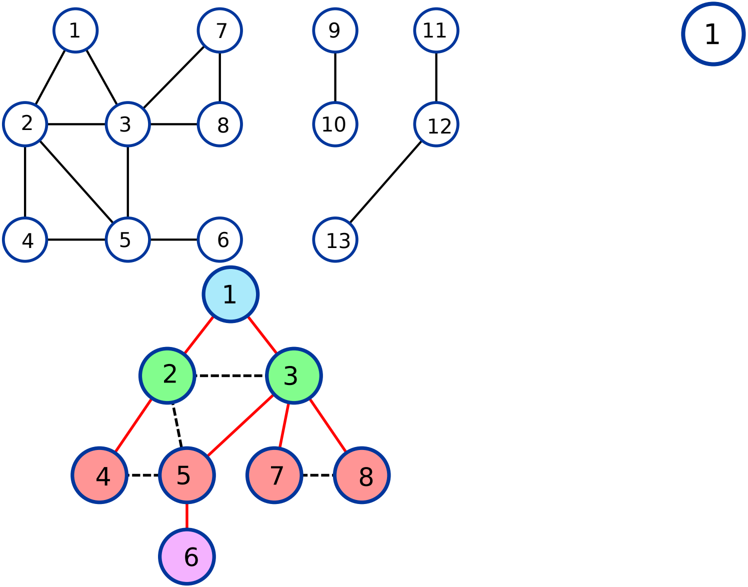

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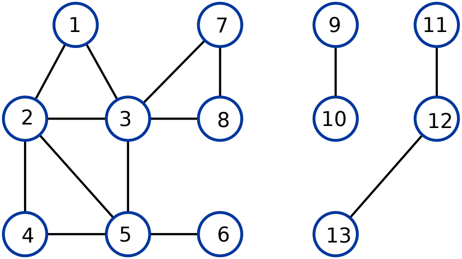


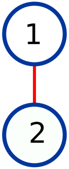


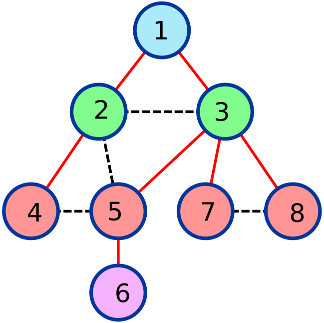

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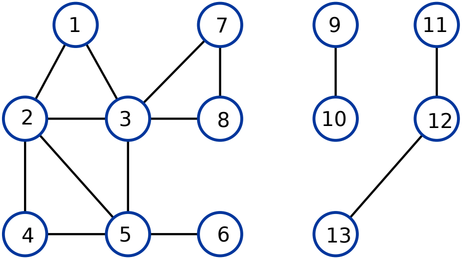


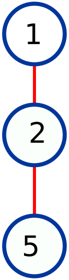


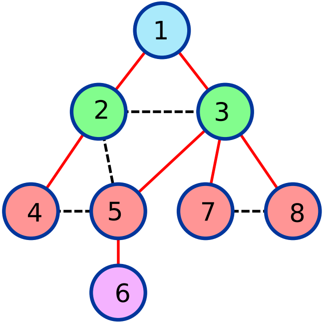

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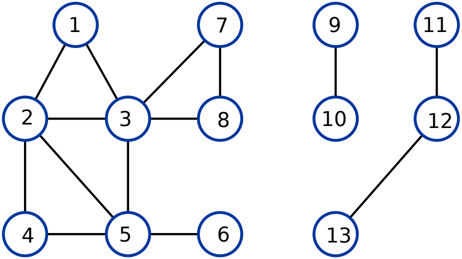


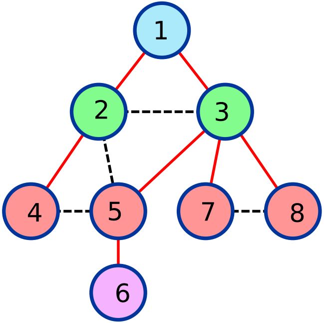


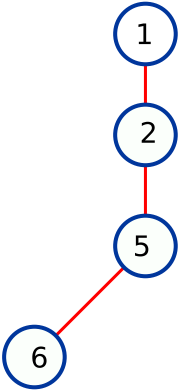

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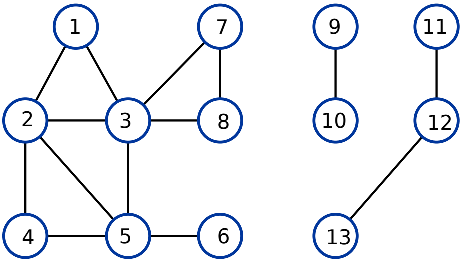


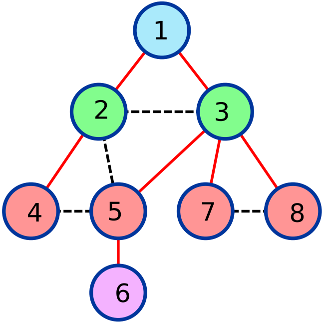


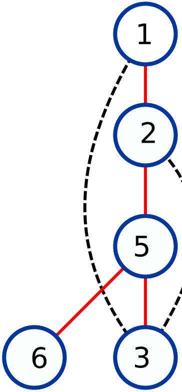

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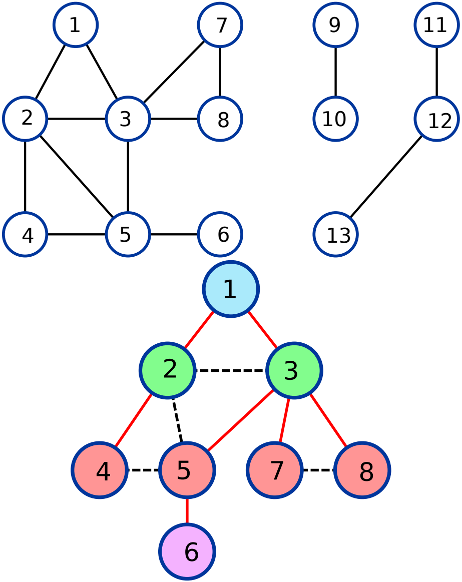


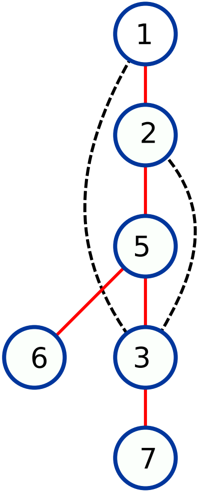

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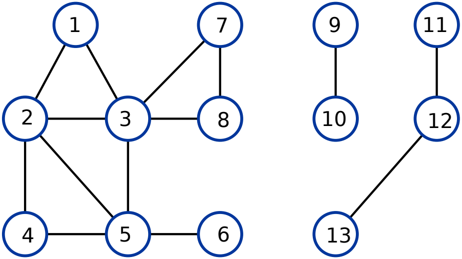


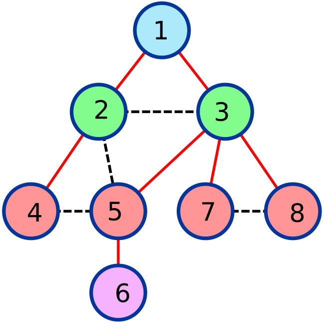


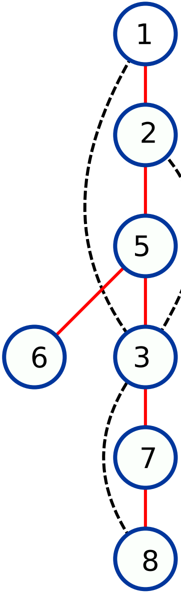

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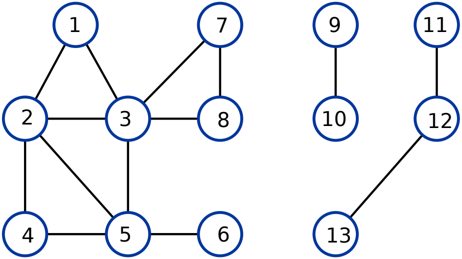


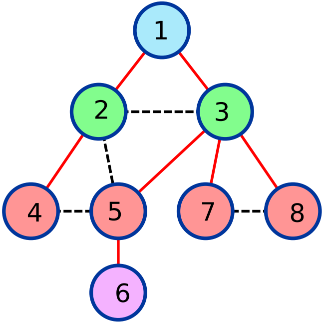


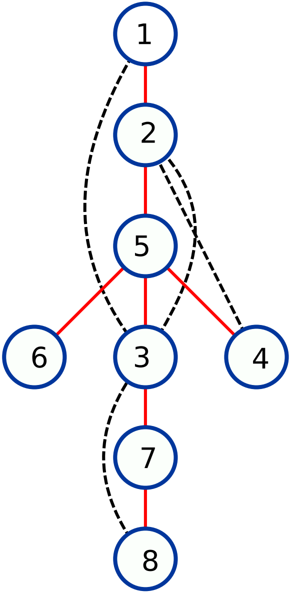

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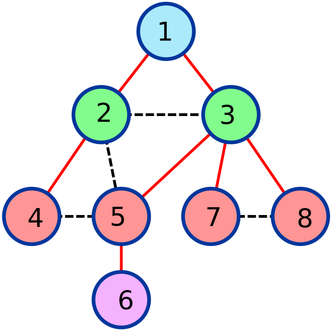


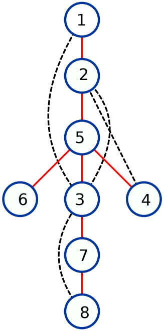


- Both visit the same set of nodes but in a different order.
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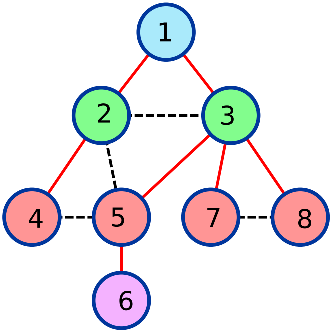


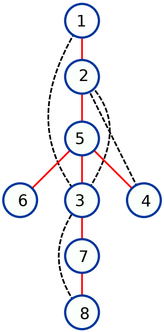


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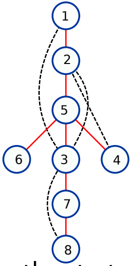


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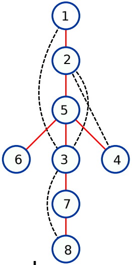


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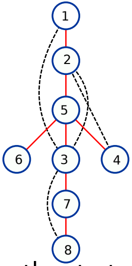


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- Proof: Assume, without loss of generality, that DFS($u$) reached $x$ first.
  - Since $(x, y)$ is an edge in $G$, it is examined during DFS($x$).
  - Since $(x, y) \notin T$, $y$ must be marked as "Explored" during DFS($x$) but before $(x, y)$ is examined.
  - Since $y$ was not marked as "Explored" before DFS($x$) was invoked, it must be marked as "Explored" between the start and the end of DFS($x$).
  - Therefore, $y$ must be a descendant of $x$ in $T$.

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# Computing All Connected Components

1. Pick an arbitrary node $s$ in $G$.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say $v$, not already visited) and repeat the BFS from $v$.
4. Repeat this process until all nodes are visited.

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- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n-1, n\}$.
- *Adjacency matrix* representation: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
  - Space used is $\Theta(n^2)$, which is optimal in the worst case.
  - Check if there is an edge between node $i$ and node $j$ in $O(1)$ time.
  - Iterate over all the edges incident on node $i$ in $\Theta(n)$ time.
- *Adjacency list* representation: array `Adj`, where `Adj`$[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in `Adj`$[u]$ and `Adj`$[v]$.
  - $n_v =$ the number of neighbours of node $v$.
  - Space used is $O(n + \sum_{v \in G} n_v) = O(n + m)$, which is optimal for every graph.
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# Data Structures for Implementation

- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
- Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.
- How do we store the set of visited nodes? Order in which we process the nodes is crucial.

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- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
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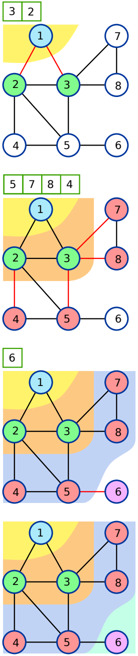
- How do we store the set of visited nodes? Order in which we process the nodes is crucial.
  - BFS: store visited nodes in a queue (first-in, first-out).
  - DFS: store visited nodes in a stack (last-in, first-out)

# Implementing BFS

- Maintain an array `Discovered` and set `Discovered`$[v]$ = *true* as soon as the algorithm sees $v$.

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = Ø
  While L[i] is not empty
    Initialize an empty list L[i + 1]
    For each node u ∈ L[i]
      Consider each edge (u, v) incident to u
      If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Add v to the list L[i + 1]
      Endif
    Endfor
    Increment the layer counter i by one
  Endwhile
```



# Using a Queue in BFS

- Instead of storing each layer in a different list, maintain all the layers in a single queue $L$.

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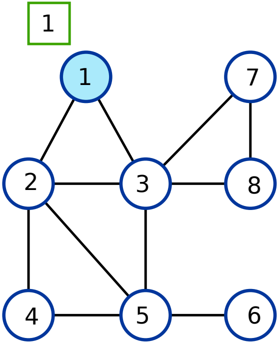
BFS(s):
   Set Discovered[s] = true
   Set Discovered[v] = false, for all other nodes v
   Initialize L to consist of the single element s
   While L is not empty
      Pop the node u at the head of L
      Consider each edge (u, v) incident on u
      If Discovered[v] = false then
         Set Discovered[v] = true
         Add edge (u, v) to the tree T
         Push v to the back of L
      Endif
   Endwhile
```



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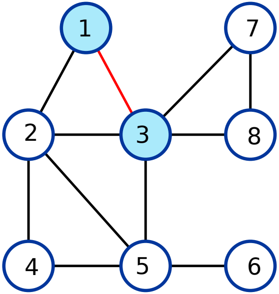

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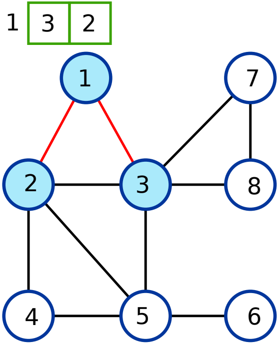

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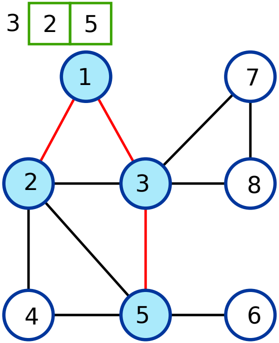

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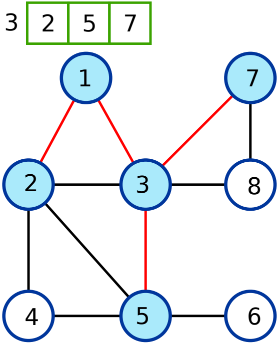

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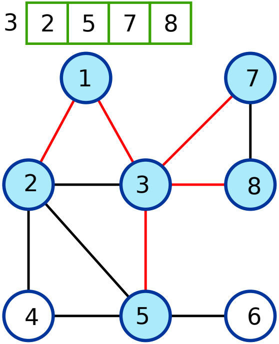

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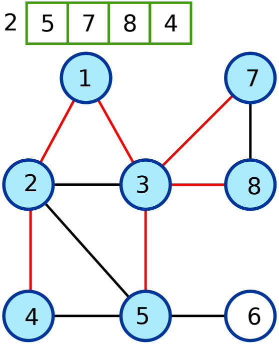

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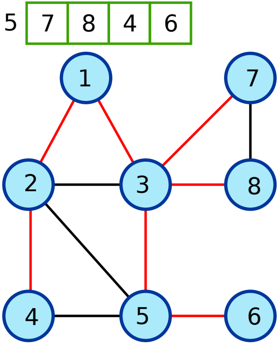

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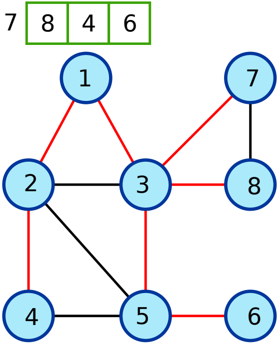

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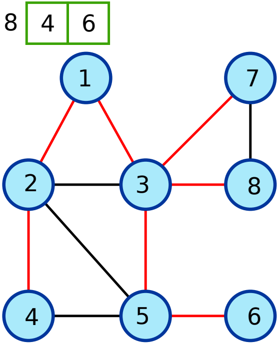

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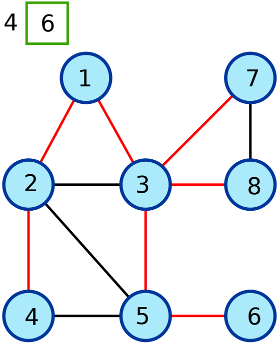

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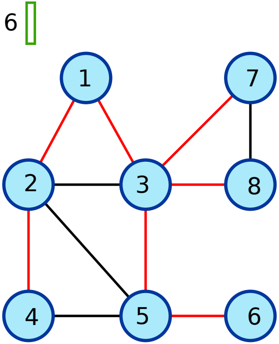

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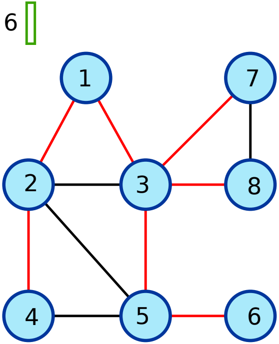


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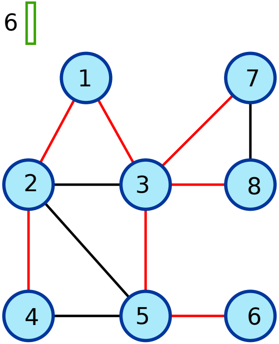


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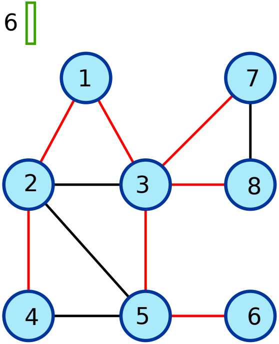


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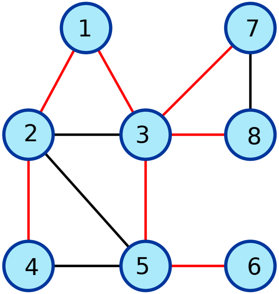


- Simple to modify this procedure to keep track of layer numbers as well. Store the pair $(u, l_u)$, where $l_u$ is the index of the layer containing $u$.
- Claim: Nodes in layer $i+1$ will appear in $L$ immediately after nodes in layer $i$. More formally: If BFS($s$) pops $(v, l_v)$ from $L$ immediately after it pops $(u, l_u)$, then either $l_v = l_u$ or $l_v = l_u + 1$.

# Analysis of BFS Implementation

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  - Total time for all `for` loops: $\sum_{u \in G} O(n_u) = O(m)$ time.
  - Maintaining layer information: $O(1)$ time per node.
  - Total time is $O(n + m)$.

# Recursive DFS

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
    If v is not marked "Explored" then
      Recursively invoke DFS(v)
    Endif
  Endfor
```

- Procedure has "tail recursion": recursive call is the last step.

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- Procedure has “tail recursion”: recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.

# Implementing DFS

- Maintain a stack $S$ to store nodes to be explored.
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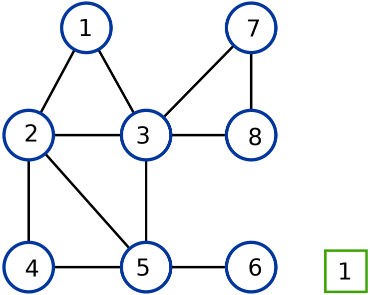

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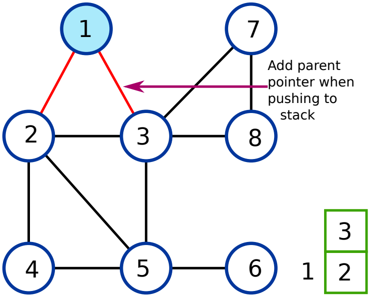

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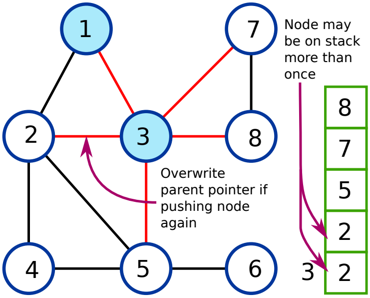

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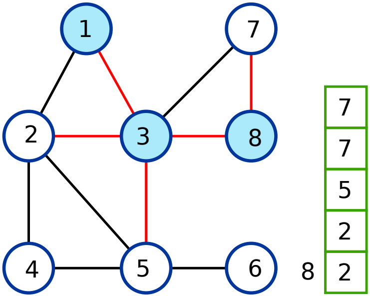

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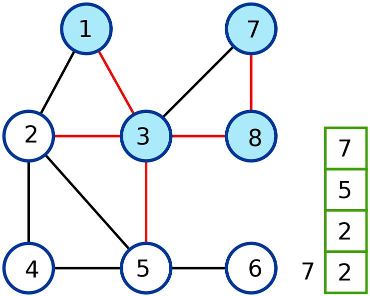

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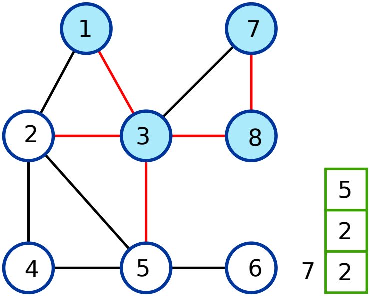

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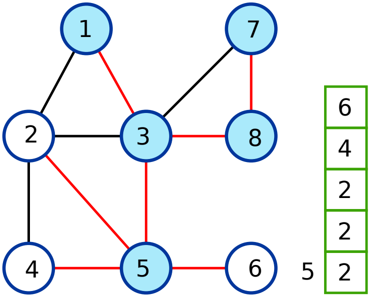

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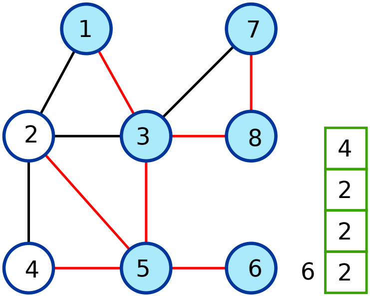

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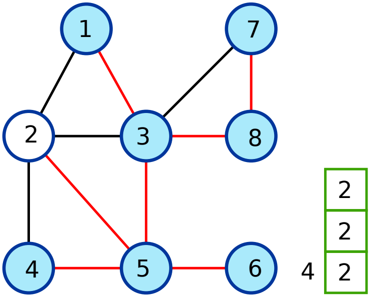

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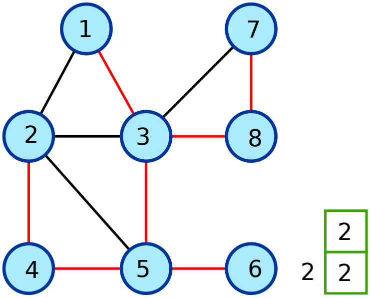

# Comparing Recursion and Iteration

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
    If v is not marked "Explored" then
      Recursively invoke DFS(v)
    Endif
  Endfor
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- The total amount of time to process edges incident on node $u$'s is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$.