Analysis of Algorithms

T. M. Murali

January 21, 2016

Force-add: Visit `https://www.cs.vt.edu/S16Force-Adds` before 3:15pm today and use the password "4104tmm$"

What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?

What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
- Goal: Develop algorithms that provably run quickly and use low amounts of space.

Worst-case Running Time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size n, as a function of n.

Worst-case Running Time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size n, as a function of n.
- Why worst-case? Why not average-case or on random inputs?

Worst-case Running Time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size n, as a function of n.
- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input.

Worst-case Running Time

- We will measure worst-case running time of an algorithm.
- Bound the largest possible running time the algorithm over all inputs of size n, as a function of n.
- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input. *Values* in the input do not matter, except for specific algorithms.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.

Polynomial Time

- Brute force algorithm: Check every possible solution.

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor c.

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor c.
- An algorithm has a *polynomial* running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size n, the running time of the algorithm is bounded by cn^d steps.

Polynomial Time

- Brute force algorithm: Check every possible solution.
- What is a brute force algorithm for sorting: given n numbers, permute them so that they appear in increasing order?
  - Try all possible $n!$ permutations of the numbers.
  - For each permutation, check if it is sorted.
  - Running time is $nn!$. Unacceptable in practice!
- Desirable scaling property: when the input size doubles, the algorithm should only slow down by some constant factor c.
- An algorithm has a *polynomial* running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size n, the running time of the algorithm is bounded by cn^d steps.

Definition

An algorithm is *efficient* if it has a polynomial running time.

Comparing Functions

- Assume all functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting:

Comparing Functions

- Assume all functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting: bubble sort, insertion sort, quick sort, merge sort, etc.

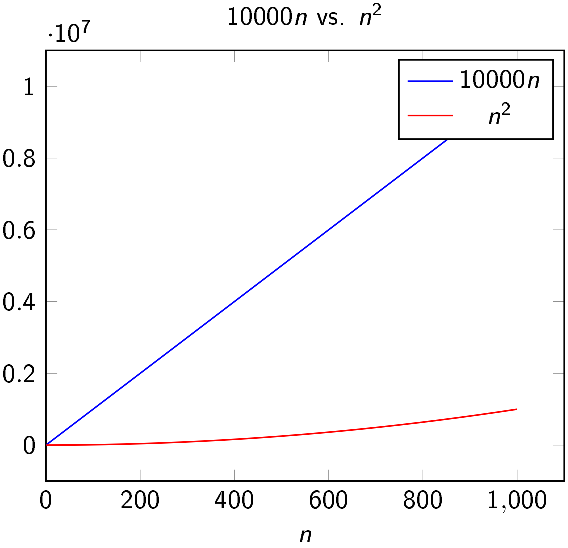
Comparing Functions

- Assume all functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting: bubble sort, insertion sort, quick sort, merge sort, etc.
- Bubble sort and insertion sort take roughly n^2 comparisons while quick sort and merge sort take roughly $n \log_2 n$ comparisons.
  - "Roughly" hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$.

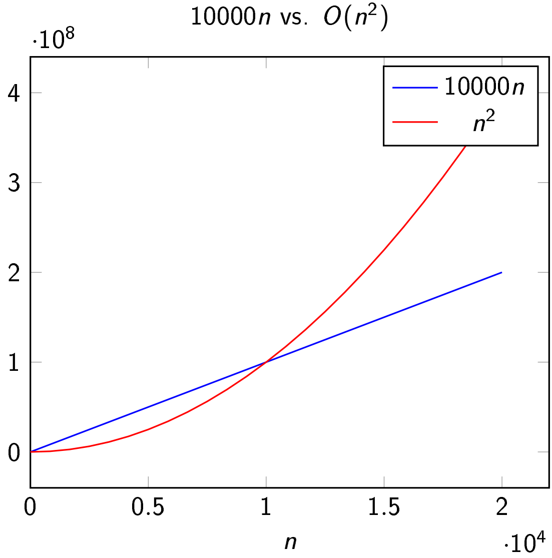
Comparing Functions

- Assume all functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting: bubble sort, insertion sort, quick sort, merge sort, etc.
- Bubble sort and insertion sort take roughly n^2 comparisons while quick sort and merge sort take roughly $n \log_2 n$ comparisons.
  - "Roughly" hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$.
- How can make statements such as the following?
  - $100n \log_2 n \leq n^2$
  - $10000n \leq n^2$
  - $5n^2 - 4n \geq 1000n \log n$

"$10000n \leq n^2$"



"$10000n \leq n^2$"



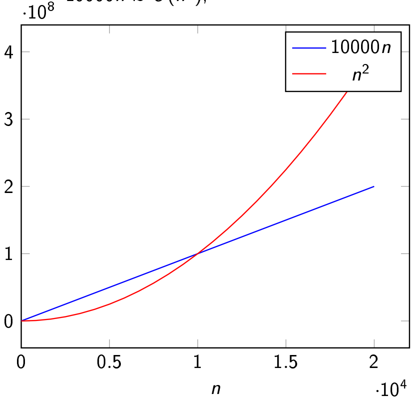
Upper Bound

Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if

for all n , we have $f(n) \leq \; g(n)$.

$10000n$ is $O(n^2)$,

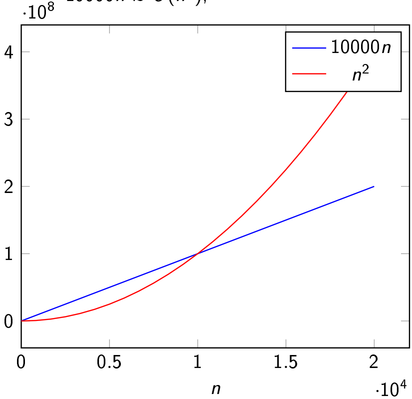


Upper Bound

Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exists constant $c > 0$ such that for all n , we have $f(n) \leq cg(n)$.

$10000n$ is $O(n^2)$,

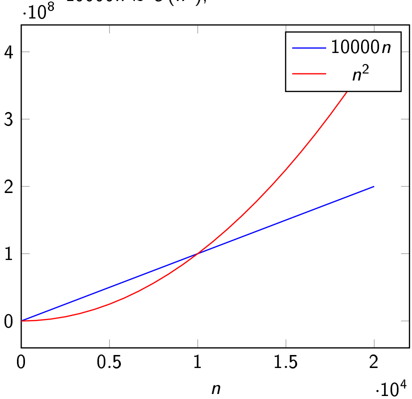


Upper Bound

Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq cg(n)$.

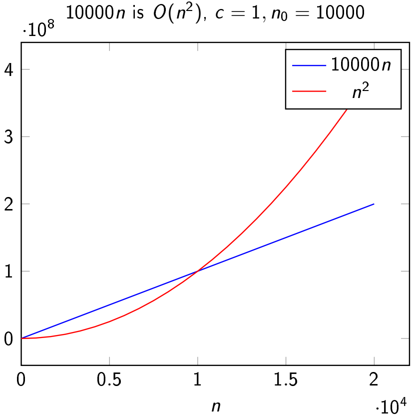
$10000n$ is $O(n^2)$,



Upper Bound

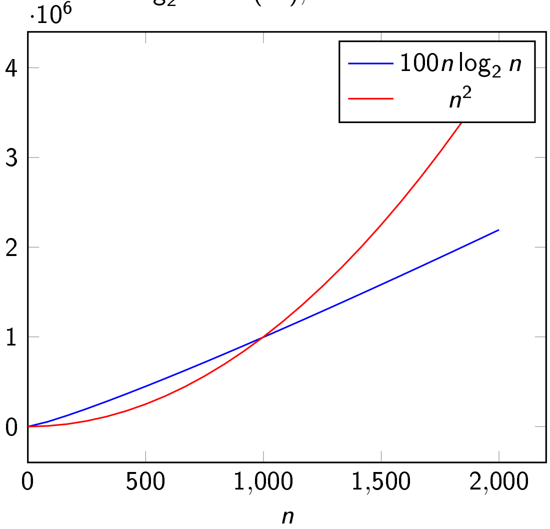
Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq cg(n)$.

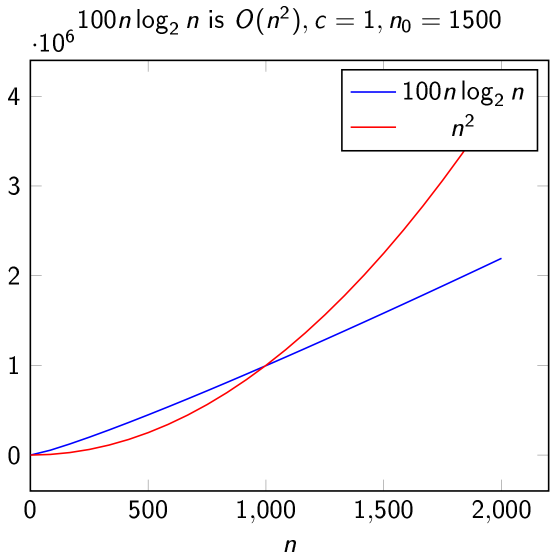


$100n\log_2 n$ and n^2

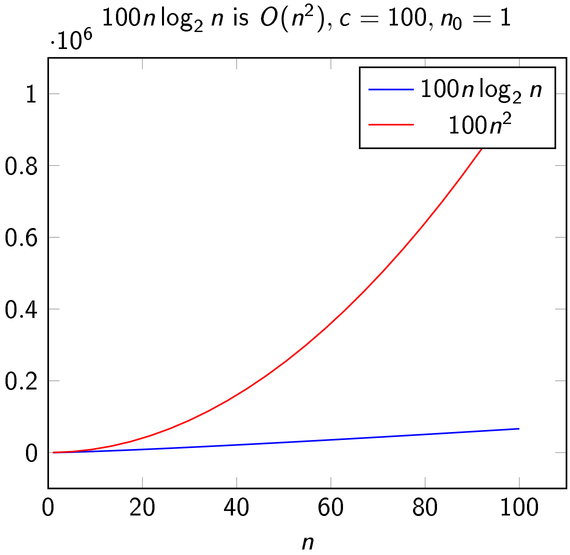
$100n\log_2 n$ is $O(n^2)$,



$100n\log_2 n$ and n^2



$100n\log_2 n$ and n^2



Lower Bound

Definition

Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if
for all n , we have $f(n) \geq \ g(n)$.

Lower Bound

Definition

Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exists constant $c > 0$ such that for all n , we have $f(n) \geq cg(n)$.

Lower Bound

Definition

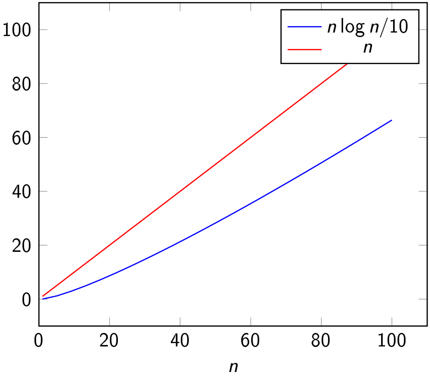
Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

Lower Bound

Definition

Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

$n \log_2 n/10$ and $\Omega(n)$

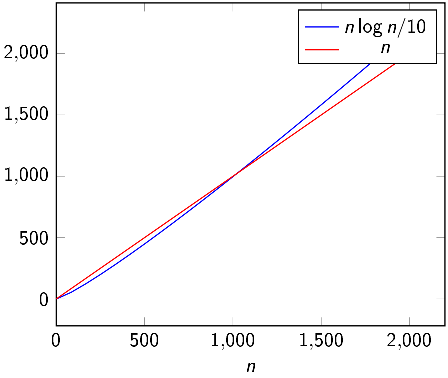


Lower Bound

Definition

Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

$n \log_2 n/10$ is $\Omega(n)$, $c = 1$, $n_0 = 1024$



Meaning of "Lower Bound" in Different Contexts

- Functions:

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$.

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$.

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$. There is some input of n numbers that will cause bubble sort to take at least $\Omega(n^2)$ time, e.g.,

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$. There is some input of n numbers that will cause bubble sort to take at least $\Omega(n^2)$ time, e.g., input the numbers in decreasing order.

Meaning of “Lower Bound” in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$. There is some input of n numbers that will cause bubble sort to take at least $\Omega(n^2)$ time, e.g., input the numbers in decreasing order.
- Problems: The problem of sorting n numbers has a lower bound of $\Omega(n \log n)$.

Meaning of "Lower Bound" in Different Contexts

- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$. There is some input of n numbers that will cause bubble sort to take at least $\Omega(n^2)$ time, e.g., input the numbers in decreasing order.
- Problems: The problem of sorting n numbers has a lower bound of $\Omega(n \log n)$. For *any* comparison-based sorting algorithm, there is at least one input for which that algorithm will take $\Omega(n \log n)$ steps.

Tight Bound

Definition

Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

Tight Bound

Definition

Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

- In all these definitions, c and n_0 are constants independent of n.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$.

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Properties of Asymptotic Growth Rates

Transitivity
- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity
- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity

- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.
- If k is a constant and there are k functions $f_i = O(h), 1 \leq i \leq k$,

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity

- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.
- If k is a constant and there are k functions $f_i = O(h), 1 \leq i \leq k$, then $f_1 + f_2 + \ldots + f_k = O(h)$.

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity

- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.
- If k is a constant and there are k functions $f_i = O(h), 1 \le i \le k$, then $f_1 + f_2 + \ldots + f_k = O(h)$.
- If $f = O(g)$, then $f + g =$

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

Additivity

- If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
- Similar statements hold for lower and tight bounds.
- If k is a constant and there are k functions $f_i = O(h), 1 \leq i \leq k$, then $f_1 + f_2 + \ldots + f_k = O(h)$.
- If $f = O(g)$, then $f + g = \Theta(g)$.

Examples

- $f(n) = pn^2 + qn + r$ is

Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.

Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?

Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \le i \le d} a_i n^i =$

Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$.
  - $O(n^d)$ is the definition of *polynomial time*.

Examples

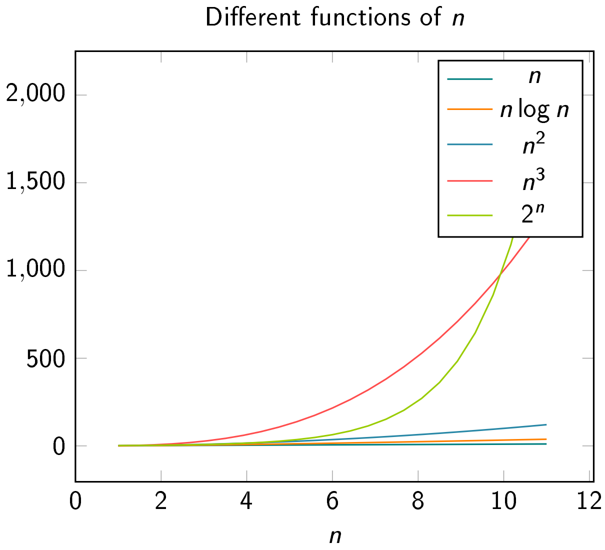
- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \le i \le d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$.
  - $O(n^d)$ is the definition of *polynomial time*.
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?

Examples

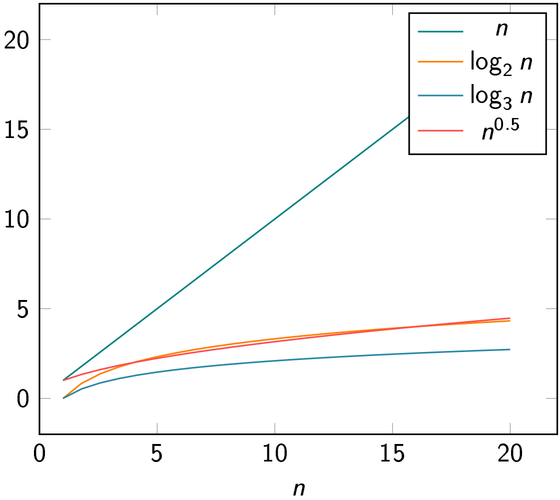
- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$.
  - $O(n^d)$ is the definition of *polynomial time*.
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every constant $x > 0$, $\log n = O(n^x)$.

Examples

- $f(n) = pn^2 + qn + r$ is $\theta(n^2)$. Can ignore lower order terms.
- Is $f(n) = pn^2 + qn + r = O(n^3)$?
- $f(n) = \sum_{0 \leq i \leq d} a_i n^i = O(n^d)$, if $d > 0$ is an integer constant and $a_d > 0$.
    - $O(n^d)$ is the definition of *polynomial time*.
- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every constant $x > 0$, $\log n = O(n^x)$.
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$.


Different functions of n
2,000
1,500
1,000
500
0
0
2
4
6
8
10
12
n
n
$n \log n$
n^2
n^3
2^n

More functions of n



Linear Time

- Running time is at most a constant factor times the size of the input.

Linear Time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.

Linear Time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list.

Linear Time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list. "Median-of-median" algorithm.
- Sub-linear time.

Linear Time

- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list. "Median-of-median" algorithm.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.

$O(n \log n)$ Time

- Any algorithm where the costliest step is sorting.

Quadratic Time

- Enumerate all pairs of elements.

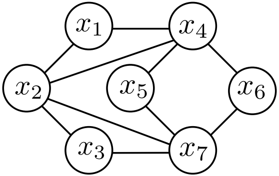
Quadratic Time

- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest.

Quadratic Time

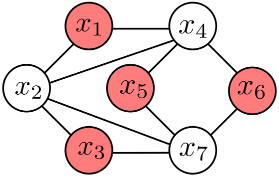
- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest. Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.

$O(n^k)$ Time



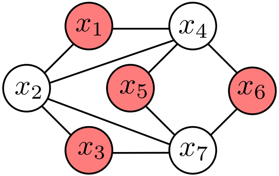
- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?

$O(n^k)$ Time



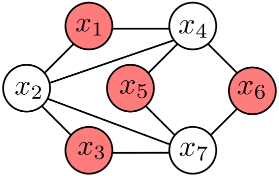
- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?

$O(n^k)$ Time



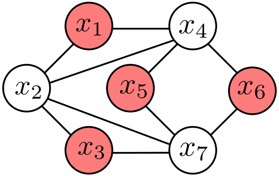
- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?
- Algorithm: For each subset S of k nodes, check if S is an independent set. If the answer is yes, report it.

$O(n^k)$ Time



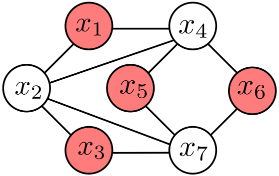
- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?
- Algorithm: For each subset S of k nodes, check if S is an independent set. If the answer is yes, report it.
- Running time is $O(k^2\binom{n}{k}) = O(n^k)$.

Beyond Polynomial Time



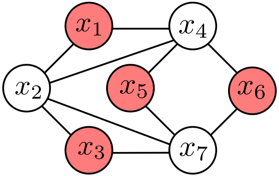
- What is the largest size of an independent set in a graph with n nodes?

Beyond Polynomial Time



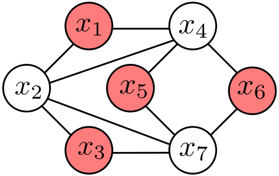
- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.

Beyond Polynomial Time



- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time?

Beyond Polynomial Time



- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time? $O(n^2 2^n)$.