Testing Bipartiteness and Dijkstra’s Algorithm

September 23, 2014
Computing All Connected Components

1. Pick an arbitrary node $s$ in $G$.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say $v$, not already visited) and repeat the BFS from $v$.
4. Repeat this process until all nodes are visited.

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- Time spent to compute each component is linear in the size of the component.
- Running time of the algorithm is \( O(m + n) \).
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- Time spent to compute each component is linear in the size of the component.
- Running time of the algorithm is linear in the total sizes of the components, i.e., $O(m + n)$. 
Bipartite Graphs

- A graph $G = (V, E)$ is *bipartite* if $V$ can be partitioned into two subsets $X$ and $Y$ such that every edge in $E$ has one endpoint in $X$ and one endpoint in $Y$.
  - $(X \times X) \cap E = \emptyset$ and $(Y \times Y) \cap E = \emptyset$.
  - Colour the nodes in $X$ red and the nodes in $Y$ blue. Then no edge in $E$ connects nodes of the same colour.

- Examples of bipartite graphs:
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- Examples of bipartite graphs: medical residents and hospitals, jobs and processors they can be scheduled on, professors and courses they can teach.

TestBipartiteness

INSTANCE: An undirected graph $G = (V, E)$

QUESTION: Is $G$ bipartite?
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Is a triangle bipartite?
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TestBipartiteness

\textbf{INSTANCE:} An undirected graph $G = (V, E)$

\textbf{QUESTION:} Is $G$ bipartite?

- Is a triangle bipartite? No.
- Generalisation: No cycle of odd length is bipartite.
- Claim: If a graph is bipartite, then it cannot contain a cycle of odd length.
Algorithm for Testing Bipartiteness

- Assume $G$ is connected. Otherwise, apply the algorithm to each connected component separately.
- Idea: Pick an arbitrary node $s$ and colour it red.
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- Idea: Pick an arbitrary node $s$ and colour it red. Colour all its neighbours blue. Colour the uncoloured neighbours of these nodes red, and so on till all nodes are coloured.
Algorithm for Testing Bipartiteness

- Assume $G$ is connected. Otherwise, apply the algorithm to each connected component separately.
- Idea: Pick an arbitrary node $s$ and colour it red. Colour all its neighbours blue. Colour the uncoloured neighbours of *these* nodes red, and so on till all nodes are coloured. Check if every edge has endpoints of different colours.

Algorithm:

1. Run BFS on $G$.
2. When we add a node $v$ to a layer $i$, set $\text{Colour}[v]$ to red if $i$ is even, otherwise to blue.
3. At the end of BFS, scan all the edges to check if there is any edge both of whose endpoints received the same colour.

Running time of this algorithm is $O(n + m)$, since we do a constant amount of work per node in addition to the time spent by BFS.
Algorithm for Testing Bipartiteness

- Assume $G$ is connected. Otherwise, apply the algorithm to each connected component separately.
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- Algorithm:
  1. Run BFS on $G$. Maintain an additional array Colour.
  2. When we add a node $v$ to a layer $i$, set Colour[$v$] to red if $i$ is even, otherwise to blue.
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▶ Algorithm:

1. Run BFS on $G$. Maintain an additional array $\text{Colour}$.
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▶ Running time of this algorithm is $O(n + m)$, since we do a constant amount of work per node in addition to the time spent by BFS.
Correctness of the Algorithm

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Correctness of the Algorithm

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2. If $G$ is not bipartite, what is the proof? The algorithm can find a cycle of odd length in $G$. 

Let $G$ be a graph and let $L_0, L_1, L_2, \ldots, L_k$ be the layers produced by BFS, starting at node $s$. Then exactly one of the following statements is true:

1. No edge of $G$ joins two nodes in the same layer: then $G$ is bipartite and nodes in even layers can be coloured red and nodes in odd layers can be coloured blue.
2. There is an edge of $G$ that joins two nodes in the same layer: then $G$ contains a cycle of odd length and cannot be bipartite.
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Shortest Path Problem

- $G(V, E)$ is a connected directed graph. Each edge $e$ has a length $l_e \geq 0$.
- $V$ has $n$ nodes and $E$ has $m$ edges.
- **Length of a path** $P$ is the sum of the lengths of the edges in $P$.
- Goal is to determine the shortest path from a specified start node $s$ to each node in $V$.
- Aside: If $G$ is undirected, convert to a directed graph by replacing each edge in $G$ by two directed edges.
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Shortest Paths

**INSTANCE:** A directed graph $G(V, E)$, a function $l : E \rightarrow R^+$, and a node $s \in V$

**SOLUTION:** A set $\{P_u, u \in V\}$, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 
Figure 4.7 A snapshot of the execution of Dijkstra’s Algorithm. The next node that will be added to the set $S$ is $x$, due to the path through $u$. 
Dijkstra’s Algorithm

- Maintain a set $S$ of explored nodes.
- For each node $u \in S$, we store a distance $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$. 
Dijkstra’s Algorithm

- Maintain a set $S$ of explored nodes.
- For each node $u \in S$, we store a distance $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$.
- For each node $v \not\in S$, we store a value $d'(v)$, which is the length of the shortest path from $s$ to $v$ using only nodes in $S$ (and $v$, of course).
- “Greedily” add a node $v$ to $S$ that is closest to $s$. 

Dijkstra’s Algorithm ($G$, $l$):

Initialize $S = \{s\}$ and $d(s) = 0$

While $S \neq V$

- For each node $x \not\in S$ compute $d'(x) = \min_{e=(u, x)}: u \in S (d(u) + l)$

- Select a node $v \not\in S$ such that $v = \arg \min_{x \not\in S} d'(x)$

- Add $v$ to $S$ and set $d(v) = d'(v)$
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Dijkstra’s Algorithm ($G, l$):

1. Initialize $S = \{s\}$ and $d(s) = 0$
2. While $S \neq V$
   - For each node $x \not\in S$ compute $d'(x) = \min_{e=(u,x): u \in S} (d(u) + l_e)$
   - Select a node $v \not\in S$ such that $v = \arg\min_{x \not\in S} d'(x)$
   - Add $v$ to $S$ and set $d(v) = d'(v)$
3. Endwhile

- $v = \arg\min_{x \not\in S} d'(x)$ means $v$ is the node that minimises the distance $d'(x)$ over all nodes $x \not\in S$. 

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While $S \neq V$
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    Select a node $v \notin S$ such that $v = \arg \min_{x \notin S} d'(x)$
    Add $v$ to $S$ and set $d(v) = d'(v)$
Endwhile

$v = \arg \min_{x \notin S} d'(x)$ means $v$ is the node that minimises the distance $d'$ over all nodes $x \notin S$.

To compute the shortest paths: when we add $v$ to $S$, store the predecessor $u$ that minimises $d'(v)$. 
Proof of Correctness

- Let $P_u$ be the shortest path computed for a node $u$.
- Claim: $P_u$ is the shortest path from $s$ to $u$.
- Prove by induction on the size of $S$. 
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  - Base case: $|S| = 1$. The only node in $S$ is $s$.
  - Inductive hypothesis:
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  - Base case: $|S| = 1$. The only node in $S$ is $s$.
  - Inductive hypothesis: $d(u)$ is correct for all nodes $u \in S$.
  - Inductive step: we add the node $v$ to $S$. Let $u$ be the $v$’s predecessor on the path $P_v$. Could there be a shorter path $P$ from $s$ to $v$?
Proof of Correctness

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Claim: \( P_u \) is the shortest path from \( s \) to \( u \).
Prove by induction on the size of \( S \).
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Inductive step: we add the node \( v \) to \( S \). Let \( u \) be the \( v \)'s predecessor on the path \( P_v \). Could there be a shorter path \( P \) from \( s \) to \( v \)?

Figure 4.8 The shortest path \( P_v \) and an alternate \( s-v \) path \( P \) through the node \( y \).
Comments about Dijkstra’s Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra’s algorithm forms a tree. Why?
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- Union of shortest paths from a fixed source $s$ forms a tree; paths not necessarily computed by Dijkstra’s algorithm.

$p_v$: shortest path from $s$ to a node $v$, $d(v)$: length of $p_v$.

- If $u$ is the second-to-last node on $p_v$, then $d(v) = d(u) + l(u, v)$.
- If $u$ precedes $w$ on $p_v$, then $d(w) = d(u) + l(u, w)$, i.e., $d(w) - d(u) = l(u, w)$.
- Suppose union of shortest paths from $s$ contains a cycle involving nodes $v_1, v_2, \ldots, v_k$ in that order around the cycle. $d(v_i) - d(v_{i-1}) = l(v_{i-1}, v_i)$, for each $2 \leq i \leq k$.$d(v_1) - d(v_k) = \sum_{i=2}^{k}(d(v_i) - d(v_{i-1})) + d(v_1) - d(v_k) = \sum_{i=2}^{k}l(v_{i-1}, v_i) + l(v_k, v_1)$.
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$\sum_{i=2}^{k} (d(v_i) - d(v_{i-1})) = \sum_{i=2}^{k} (d(v_i) - d(v_{i-1})) + \sum_{i=2}^{k} l(v_{i-1}, v_i)$
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\begin{align*}
  d(v_i) - d(v_{i-1}) &= l(v_{i-1}, v_i), \text{ for each } 2 \leq i \leq k \\
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\sum_{i=2}^{k} (d(v_i) - d(v_{i-1})) + d(v_1) - d(v_k) = \sum_{i=2}^{k} l(v_{i-1}, v_i) + l(v_k, v_1)
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\sum_{i=2}^{k} (d(v_i) - d(v_{i-1})) + d(v_1) - d(v_k) = \sum_{i=2}^{k} l(v_{i-1}, v_i) + l(v_k, v_1)
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0 = \sum_{i=2}^{k} l(v_{i-1}, v_i) + l(v_k, v_1)
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Implementing Dijkstra’s Algorithm

Dijkstra’s Algorithm($G, l$):
Initialize $S = \{s\}$ and $d(s) = 0$
While $S \neq V$
    For each node $x \notin S$ compute $d'(x) = \min_{e=(u,x): u \in S} (d(u) + l_e)$
    Select a node $v \notin S$ such that $v = \arg \min_{x \notin S} d'(x)$
    Add $v$ to $S$ and set $d(v) = d'(v)$
Endwhile

▶ How many iterations are there of the while loop?
Implementing Dijkstra’s Algorithm

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Initialize $S = \{s\}$ and $d(s) = 0$
While $S \neq V$
    For each node $x \notin S$ compute $d'(x) = \min_{e=(u,x): u \in S} (d(u) + l_e)$
    Select a node $v \notin S$ such that $v = \arg \min_{x \notin S} d'(x)$
    Add $v$ to $S$ and set $d(v) = d'(v)$
Endwhile

▶ How many iterations are there of the while loop? $n - 1$. 
Implementing Dijkstra’s Algorithm

Dijkstra’s Algorithm\((G, l)\):

1. Initialize \( S = \{s\} \) and \( d(s) = 0 \)
2. While \( S \neq V \)
   1. For each node \( x \notin S \) compute \( d'(x) = \min_{e = (u, x) : u \in S}(d(u) + l_e) \)
   2. Select a node \( v \notin S \) such that \( v = \arg \min_{x \notin S} d'(x) \)
   3. Add \( v \) to \( S \) and set \( d(v) = d'(v) \)

How many iterations are there of the while loop? \( n - 1 \).

In each iteration, for each node \( x \notin S \), compute

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Initialize $S = \{s\}$ and $d(s) = 0$
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Endwhile

- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $x \not\in S$, compute
  $$d'(x) = \min_{e = (u, x), u \in S} d(u) + l_e$$
- Running time per iteration is $O(m)$.
Implementing Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l)\):

Initialize \(S = \{s\} \) and \(d(s) = 0\)

While \(S \neq V\)

For each node \(x \notin S\) compute \(d'(x) = \min_{e=(u,x): u \in S} (d(u) + l_e)\)

Select a node \(v \notin S\) such that \(v = \arg \min_{x \notin S} d'(x)\)

Add \(v\) to \(S\) and set \(d(v) = d'(v)\)

Endwhile

- How many iterations are there of the while loop? \(n - 1\).
- In each iteration, for each node \(x \notin S\), compute

\[
  d'(x) = \min_{e=(u,x), u \in S} d(u) + l_e
\]

- Running time per iteration is \(O(m)\), yielding an overall running time of \(O(nm)\).
A Faster implementation of Dijkstra’s Algorithm

Dijkstra’s Algorithm($G, l$):
Initialize $S = \{s\}$ and $d(s) = 0$
While $S \neq V$
    For each node $x \notin S$ compute $d'(x) = \min_{e=(u,x): u \in S} (d(u) + l_e)$
    Select a node $v \notin S$ such that $v = \arg\min_{x \notin S} d'(x)$
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Endwhile

▶ Observation: If we add $v$ to $S$, $d'(x)$ changes only if $(v, x)$ is an edge in $G$. 

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A Faster implementation of Dijkstra’s Algorithm

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Initialize $S = \{s\}$ and $d(s) = 0$
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Endwhile

- Observation: If we add $v$ to $S$, $d'(x)$ changes only if $(v, x)$ is an edge in $G$.
- Store the pairs $(x, d'(x))$ for each node $x \in V - S$ in a priority queue, with $d'(x)$ as the key.
- Determine the next node $v$ to add to $S$ using ExtractMin.
- After adding $v$ to $S$, for each node $w$ such that $(v, w)$ is an edge in $G$, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$,
  1. Set $d'(w) = d(v) + l_{(v,w)}$.
  2. Update $w$’s key to the new value of $d'(w)$ using ChangeKey.

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A Faster implementation of Dijkstra’s Algorithm

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For each node \( x \notin S \) compute \( d'(x) = \min_{e=(u,x) : u \in S} (d(u) + l_e) \)

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- How many times are ExtractMin and ChangeKey invoked?
A Faster implementation of Dijkstra’s Algorithm

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While \(S \neq V\)

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Select a node \(v \notin S\) such that \(v = \arg \min_{x \notin S} d'(x)\)

Add \(v\) to \(S\) and set \(d(v) = d'(v)\)

Endwhile

- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v, x)\) is an edge in \(G\).
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- How many times are ExtractMin and ChangeKey invoked? \(n - 1\) and \(m\)
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How many times are ExtractMin and ChangeKey invoked? $n - 1$ and $m$.
Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm($G, l$):

Initialize $S = \{s\}$ and $d(s) = 0$

Insert $(s, 0)$ into a priority queue $Q$.

While $S \neq V$

$(v, k) = \text{EXTRACTMIN}(Q)$

Add $v$ to $S$ and set $d(v) = k$

For each node $w$ such that $e = (v, w)$ is an edge in $G$

If $d(v) + l_e < d'(w)$,

Set $d'(w) = d(v) + l_e$

CHANGEKEY($Q, w, d'(w)$).

Endwhile