Analysis of Algorithms

August 28, 2014

Problem Example

Find Minimum

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: Pair (i, x_i) such that $x_i = \min\{x_j \mid 1 \leq j \leq n\}$.

Algorithm Example

Find-Minimum$(x_1, x_2, \ldots, x_n)$

$i \leftarrow 1$

for $j \leftarrow 2$ **to** n

  do if $x_j < x_i$

    then $i \leftarrow j$

return (i, x_i)

Running Time of Algorithm

Find-Minimum$(x_1, x_2, \ldots, x_n)$

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for j ← 2 to n
    do if x_j < x_i
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- At most $2n - 1$ assignments and $n - 1$ comparisons.

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- Is $a < c$?

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- Is $a < c$? No. Since the algorithm returns (a, x_a), $x_a \leq x_j$, for all $a < j \leq n$. Therefore $c < a$.
- What does the algorithm do when $j = c$? *It must set i to c*, since we have been told that x_c is the smallest element.
- What does the algorithm do when $j = a$ (which happens *after* $j = c$)? Since $x_c < x_a$, the value of i does not change.
- Therefore, the algorithm does not return (a, x_a) yielding a contradiction.

Correctness of Algorithm: Proof 2

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- Proof by induction: What is true at the end of each iteration?

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- Claim is true

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- Claim is true $\Rightarrow$ algorithm is correct (set $j = n$).

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- Proof of the claim involves three steps.

1. Base case: $j = 1$ (before loop). $x_i = \min\{x_m \mid 1 \leq m \leq 1\}$ is trivially true.
2. Inductive hypothesis: Assume $x_i = \min\{x_m \mid 1 \leq m \leq j\}$.
3. Inductive step: Prove $x_i = \min\{x_m \mid 1 \leq m \leq j + 1\}$.
   - In the loop, i is set to be $j + 1$ if and only if $x_{j+1} < x_i$.
   - Therefore, x_i is the smallest of $x_1, x_2, \ldots, x_{j+1}$ when the loop ends.

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- Why does this strategy work?

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$$\begin{aligned} P(k+1) &= \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)(\frac{k}{2} + 2) = \frac{(k+1)(k+2)}{2}. \end{aligned}$$

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Given

$$P(n) = \begin{cases} P(\lfloor \frac{n}{2} \rfloor) + 1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

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- Inductive step: $P(k+1) = P(\lfloor \frac{k+1}{2} \rfloor) + 1$.
- We are stuck since inductive hypothesis does not say anything about $P(\lfloor \frac{k+1}{2} \rfloor)$.

Strong Induction

- Use strong induction: In the inductive hypothesis, assume that $P(i)$ is true for all $i \leq k$.

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$$
\begin{aligned}
P(k+1) &= P(\lfloor \frac{k+1}{2} \rfloor) + 1 \\
&\leq 1 + \log_2(\lfloor \frac{k+1}{2} \rfloor) + 1 \\
&\leq 1 + \log_2(k+1) - 1 + 1 = 1 + \log_2(k+1)
\end{aligned}
$$

Efficiency

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?

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- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
- Goal: Develop algorithms that provably run quickly and/or use low amounts of space.

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- Bound the largest possible running time the algorithm over all inputs of size n, as a function of n.
- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input. *Values* in the input do not matter, except for specific algorithms.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.

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Definition

An algorithm is *tractable* if it has a polynomial running time.

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- Bubble sort and insertion sort take roughly n^2 comparisons while quick sort and merge sort take roughly $n \log_2 n$ comparisons.
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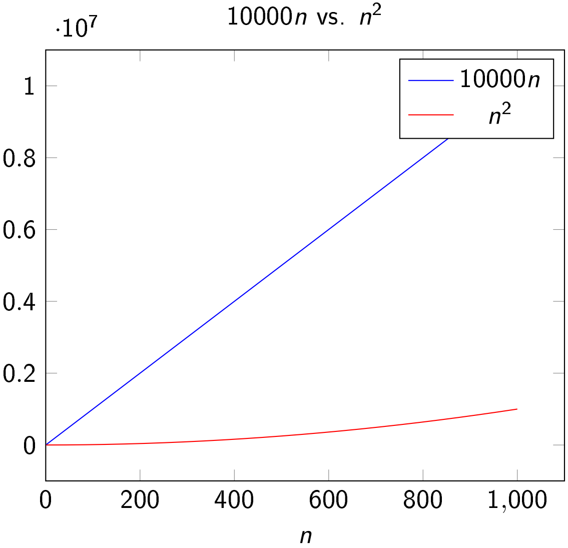
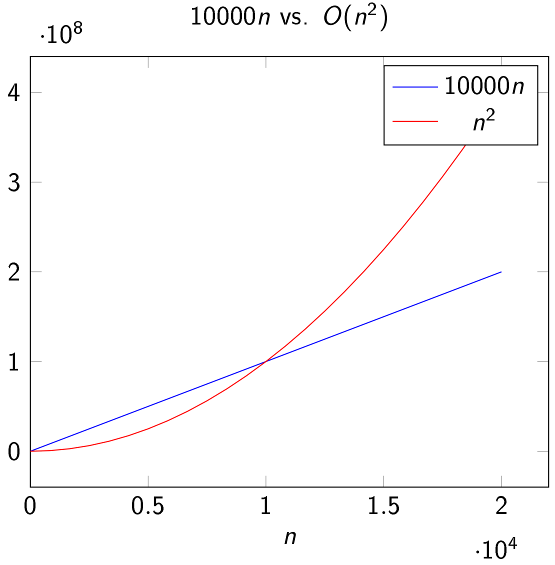
  - "Roughly" hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$.
- How can make statements such as the following?
  - "$100n \log_2 n \leq n^2$"
  - "$10000n \leq n^2$"
  - "$5n^2 - 4n \geq 1000n \log n$"

"$10000n \leq n^2$"



"$10000n \leq n^2$"



Upper Bound

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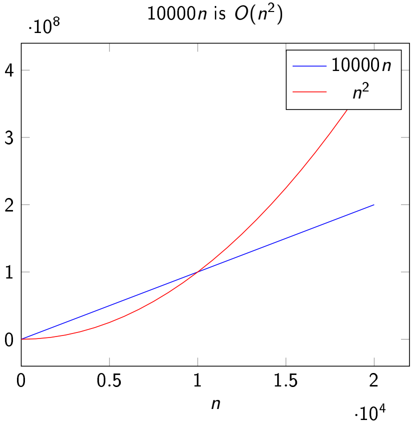
Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if
for all n , we have $f(n) \leq \ g(n)$.



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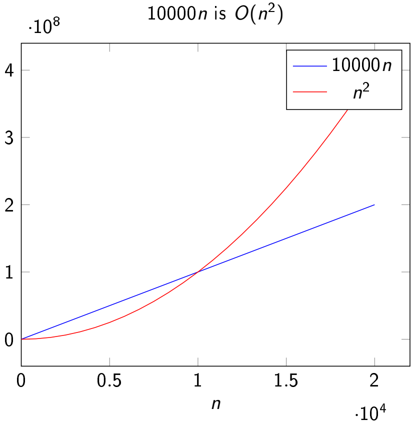

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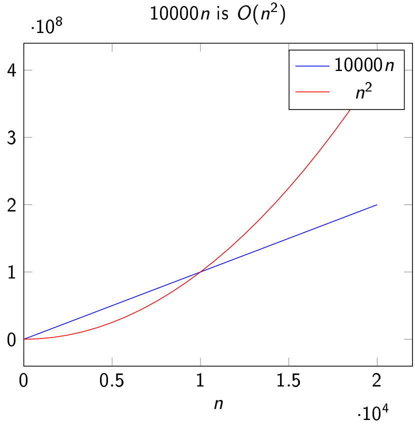
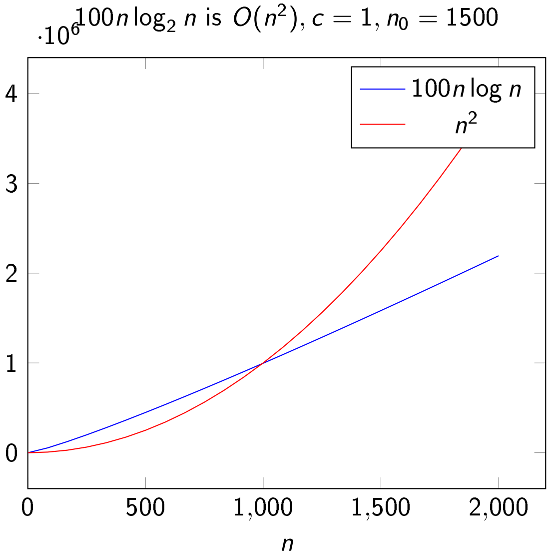


$100n \log_2 n$ and n^2



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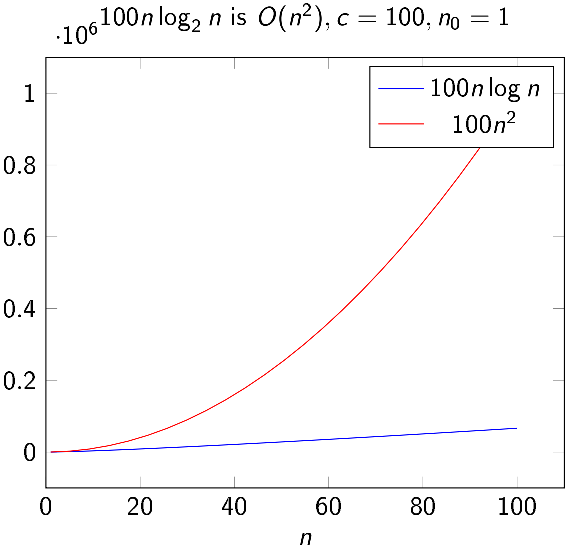

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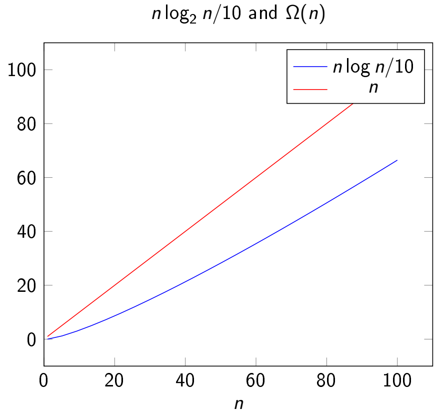

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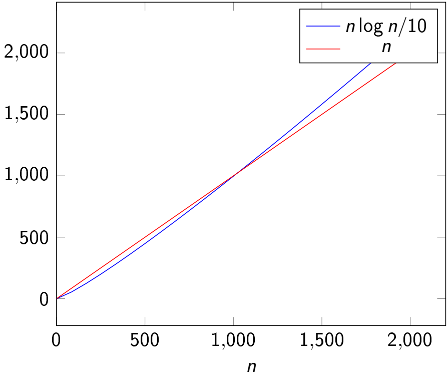
$n \log_2 n/10$ is $\Omega(n)$, $c = 1$, $n_0 = 1024$



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- Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$.

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- Functions: n is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. This statement is purely about these two mathematical functions without relevance to any algorithm or problem.
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- Problems: The problem of sorting n numbers has a lower bound of $\Omega(n \log n)$. For *any* comparison-based sorting algorithm, there is at least one input for which that algorithm will take $\Omega(n \log n)$ steps.

Tight Bound

Definition

Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

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- In all these definitions, c and n_0 are constants independent of n.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$.

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
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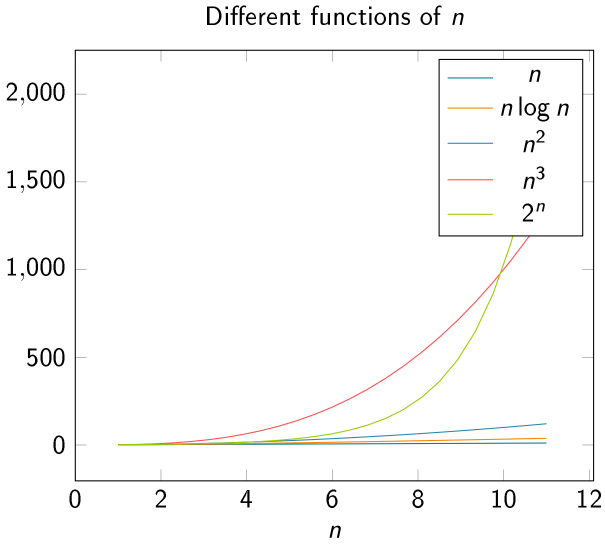
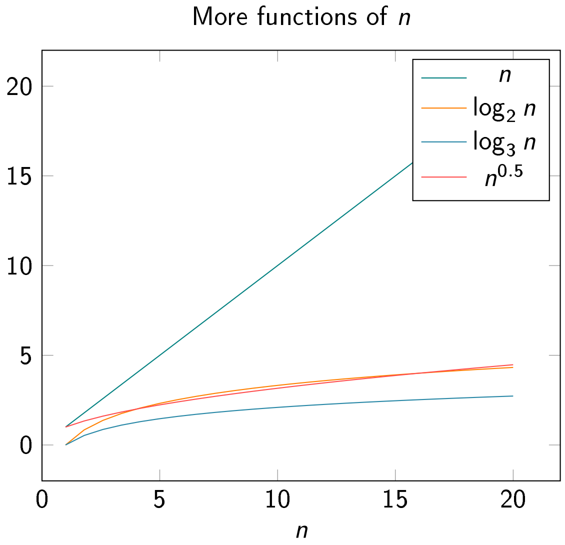
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$.


Different functions of n
2,000
1,500
1,000
500
0
0
2
4
6
8
10
12
n
n
n log n
n2
n3
2n


More functions of n
n
log2 n
log3 n
n0.5
0
5
10
15
20
0
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20
n

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- Finding the minimum, merging two sorted lists.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.

$O(n \log n)$ Time

- Any algorithm where the costliest step is sorting.

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- Given a set of n points in the plane, find the pair that are the closest. Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.

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- Running time is $O(k^2 \binom{n}{k}) = O(n^k)$.

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- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time?

Beyond Polynomial Time

- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time? $O(n^2 2^n)$.