Final Examination

CS 4104 (Spring 2014)

Assigned: May 5, 2014.
Due: on Scholar by 5pm on May 13, 2014.

Name: ____________________

9-digit PID: ____________________

Instructions

1. **You must work on your own for this examination, i.e., you are not allowed to have a partner.**

2. For every algorithm you describe, prove its correctness, and state and prove the running time of the algorithm. I am looking for clear descriptions of algorithms and for the most efficient algorithms and analysis that you can come up with. I am not specifying the desired running time for each algorithm. I will give partial credit to non-optimal algorithms, as long as they are correct.

3. If you prove that a problem is $\mathcal{NP}$-Complete, remember to state how long the certificate is, how long it takes to check that the certificate is correct, and what the running time of the transformation is. All you need to show is that the transformation can be performed in polynomial time. Your transformation need not be the most efficient possible.

4. You may consult the textbook, your notes, or the course web site to solve the problems in the examination. Of course, the TAs and I are available to answer your questions. You **may not** work on the exam with anyone else, ask anyone questions, or consult other textbooks or sites on the Web for answers. **Do not use** concepts from chapters in the textbook that we have not covered, i.e., Chapters 12 and 13.

5. You must prepare your solutions digitally, i.e., do not hand-write your solutions.

6. I prefer that you use LaTeX to prepare your solutions. However, I will not penalise you if you use a different system. To use LaTeX, you may find it convenient to download the LaTeX source file for this document from the link on the course web site. At the end of each problem are three commented lines that look like this:

   ```
   % \solution{
   %
   % }
   ```

   You can uncomment these lines and type in your solution within the curly braces.

Good luck!

Problem 1 (10 points) Let us start with some quickies. For each statement below, say whether it is true or false or fill in the blank. You do not need to provide a rationale for your solution.

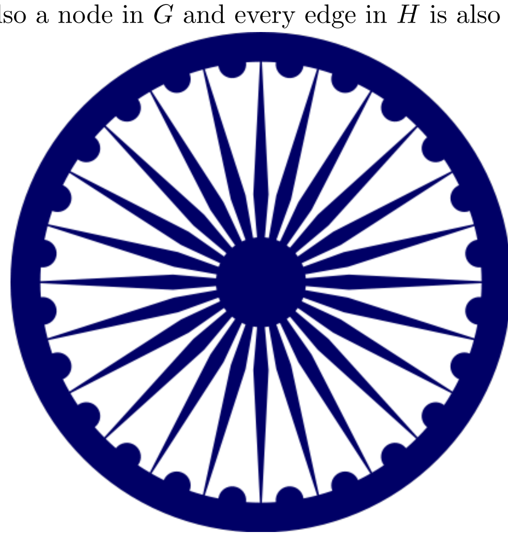
1. If A and B are problems such that $B \in \mathcal{P}$ and A $\leq_P$ B, then $A \in \mathcal{P}$.
2. The costliest edge in a directed graph can never be an edge in a minimum spanning tree of the graph.
3. An algorithm takes an input of size n, performs some computation in $O(n)$ time, and then recurses on an input of size $n/2$ to compute the solution. The running time of this algorithm is $O(n)$.
4. ______'_ celebrates its 20-year anniversary on May 7, 2014. *Hint:* A Blacksburg insitution.
5. In the original comics, Spiderman (or The *Amazing* Spiderman, as he is called these days) got his powers after being bitten by a ___________ spider.

Problem 2 (15 points) Consider the residual graph G_f when the Ford-Fulkerson algorithm terminates; we have used the notation $\nu(f)$ to denote the value of this flow. Let $rc(e)$ denote the residual capacity of an edge e in G_f. Let A^* be the set of all nodes reachable from s in G_f (by a directed path of one or more edges) and $B^* = V - A^*$. Consider the set of edges (u, v) where u is a node in B^* and v is a node in A^*. What is the total residual capacity of these edges, i.e., what is the value of

$$\sum_{(u,v) \in G_f, u \in B^*, v \in A*} rc(e)?$$

Provide a brief explanation.

Problem 3 (20 points) The flag of a certain populous country contains a symbol called the “Ashoka Chakra” (see the image below). This symbol has a central hub and 24 spokes. Naturally, this reminds us of a graph with 25 nodes and 48 edges, of which 24 nodes are connected by a cycle, and the 25th node is connected to each of the other 24 nodes. A *generalised k-chakra* is a graph with $k+1$ nodes and $2k$ edges such that k nodes lie on a cycle and the $k+1$st node is connected to each of the other k nodes. Given an undirected graph G and an integer k, prove that the problem of determining if G contains a generalised k-chakra as a subgraph is $\mathcal{NP}$-Complete. (We say that a graph H is a *subgraph* of a graph G if every node in H is also a node in G and every edge in H is also an edge in G.)



I will get you started on the solution. Proving that k-chakra is in $\mathcal{NP}$ is easy. A certificate is just a candidate k-chakra. The certifier checks that this chakra contains $k+1$ nodes and $2k$ nodes in the right configuration. The certificate takes $O(k)$ time to run.

Let us move on to proving that some $\mathcal{NP}$-Complete problem is reducible to the k-chakra problem. Suppose G has n nodes. Let us consider the special case that $k = n$. An n-chakra looks suspiciously like a Hamiltonian cycle, except that the chakra has more edges. Let us reduce Hamiltonian cycle in undirected graphs (which we know to be $\mathcal{NP}$-Complete) to the k-chakra problem. Suppose H is an undirected graph that is an input to the Hamiltonian cycle problem. We want to convert it to a graph G that will be input to the k-chakra problem such that H contains a Hamiltonian cycle iff G contains a n-chakra. To complete the reduction, answer the following three questions:

(a) (5 points) Describe how you will convert an arbitrary undirected graph H that is input to the Hamiltonian cycle problem into an undirected graph G that is an input for the chakra problem.

(b) (5 points) If H contains a Hamiltonian cycle, prove that G contains a n-chakra.

(c) (10 points) If G contains a n-chakra, prove that H contains a Hamiltonian cycle. *Note:* there is a subtlety here that you have to be careful about.

Problem 4 (10 points) Prove the following statement. In a flow network where the maximum flow has value f, there must exist a path from s-t with capacity at least f/m, where m is the number of edges in the graph. By the *capacity* of an s-t path, we mean the smallest capacity of an edge in that path. *Hint:* Use a proof by contradiction. Suppose every s-t path has at least one edge with capacity less than f/m. What can you say about the set of all these edges?

Problem 5 (20 points) After graduation, you start a job at a company that is building a space elevator! People travelling to space in the elevator need to eat. Therefore, your company is planning to open a series of restaurants along the elevator. There are n potential locations for these restaurants. Starting from the beginning of the space elevator, these locations are, in miles and in increasing order, $m_1, m_2, m_3, \ldots m_{n-1}, m_n$. Your company wants to minimize the costs of constructing and maintaining these restaurants as well as maximising its profits. It devises the following rules:

(i) At each location, it can open at most one restaurant.

(ii) For each $1 \leq i \leq m$, the expected profit from opening a restaurant at location i is $p_i > 0$.

(iii) Any two restaurants your company builds should be at least k miles apart, where k is a positive integer.

Since the company hired you because you have taken CS 4104, it asks you to devise an efficient algorithm to decide where to build the restaurants so as to compute the maximum total profit subject to the given rules. Use your advanced knowledge of algorithms to impress your bosses and peers and collect a hefty Christmas bonus!

Problem 6 (10 points) The Department of Computer Science at Virginia Tech needs to teach a set C of n courses in Fall 2014. There are m professors in the department. Each course can be taught by exactly two professors. A single professor can teach anywhere from 1 to all n courses. By "can teach", I mean that the professor has the ability and expertise to teach the course. The Department has a budget to pay at most k professors in Fall 2014, where k is an integer between 1 and m. It seeks to determine if there are at most k professors who can teach all n courses among themselves. Prove that this problem is $\mathcal{NP}$-Complete. *Hint:* This problem is easier than it looks.

Problem 7 (15 points) In a directed graph, let π be the second shortest path between two nodes s and t in the graph. To be precise, the length of π is larger than the the shortest path length from s to t but less than or equal to the length of all other s-t paths. Let v be a node on this path. We will use the $\pi(s, v)$ to denote the sub-path of π from s to v and $\pi(v, t)$ to denote the sub-path of π from v to t.

(a) (10 points) Prove that either $\pi(s, v)$ is the shortest path from s to v or $\pi(v, t)$ is the shortest path from v to t.

(b) (5 points) Suppose that v is a node such that $\pi(v, t)$ is the shortest path from v to t. What can you say about $\pi(s, v)$? I just want a single sentence answer.