Greedy Graph Algorithms

T. M. Murali

September 16, 21, 23, and 28, 2009
Shortest Path Problem

- \( G(V, E) \) is a connected directed graph. Each edge \( e \) has a length \( l_e \geq 0 \).
- \( V \) has \( n \) nodes and \( E \) has \( m \) edges.
- **Length of a path** \( P \) is the sum of the lengths of the edges in \( P \).
- Goal is to determine the shortest path from a specified start node \( s \) to each node in \( V \).
- Aside: If \( G \) is undirected, convert to a directed graph by replacing each edge in \( G \) by two directed edges.
Shortest Path Problem

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- $V$ has $n$ nodes and $E$ has $m$ edges.
- **Length of a path** $P$ is the sum of the lengths of the edges in $P$.
- Goal is to determine the shortest path from a specified start node $s$ to each node in $V$.
- Aside: If $G$ is undirected, convert to a directed graph by replacing each edge in $G$ by two directed edges.

**Shortest Paths**

**INSTANCE:** A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

**SOLUTION:** A set $\{P_u, u \in V\}$, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 
Figure 4.7 A snapshot of the execution of Dijkstra’s Algorithm. The next node that will be added to the set $S$ is $x$, due to the path through $u$. 
Dijkstra’s Algorithm

- Maintain a set $S$ of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from $s$ to $u$.
- “Greedily” add a node $v$ to $S$ that is closest to $s$. 

\[ d'(v) = \text{length of shortest path from } s \text{ to } v \text{ using only nodes in } S \]
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**Dijkstra's Algorithm \((G, \ell)\)**

Let \( S \) be the set of explored nodes

- For each \( u \in S \), we store a distance \( d(u) \)

Initially \( S = \{s\} \) and \( d(s) = 0 \)

While \( S \neq V \)

- Select a node \( v \notin S \) with at least one edge from \( S \) for which \( d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e \) is as small as possible

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

EndWhile
Dijkstra’s Algorithm

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$$d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$$

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Add $v$ to $S$ and define $d(v) = d'(v)$

EndWhile

- $d'(v) = \text{length of shortest path from } s \text{ to } v \text{ using only nodes in } S$.
- To compute the shortest paths:
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$$d'(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e$$

is as small as possible

Add $v$ to $S$ and define $d(v) = d'(v)$

EndWhile

- $d'(v) =$ length of shortest path from $s$ to $v$ using only nodes in $S$.
- To compute the shortest paths: store the predecessor $u$ that minimises $d'(v)$. 
Proof of Correctness

- Let $P_u$ be the path computed for a node $u$.
- Claim: $P_u$ is the shortest path from $s$ to $u$.
- Prove by induction on the size of $S$. 
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- Claim: $P_u$ is the shortest path from $s$ to $u$.
- Prove by induction on the size of $S$.
  - Base case: $|S| = 1$. The only node in $S$ is $s$.
  - Inductive step: we add the node $v$ to $S$. Let $u$ be the $v$’s predecessor on the path $P_v$. Could there be a shorter path $P$ from $s$ to $v$?
Proof of Correctness

▶ Let $P_u$ be the path computed for a node $u$.
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  ▶ Base case: $|S| = 1$. The only node in $S$ is $s$.
  ▶ Inductive step: we add the node $v$ to $S$. Let $u$ be the $v$’s predecessor on the path $P_v$. Could there be a shorter path $P$ from $s$ to $v$?

**Figure 4.8** The shortest path $P_v$ and an alternate $s$-$v$ path $P$ through the node $y$. 
Comments about Dijkstra’s Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output form a tree. Why?
Implementing Dijkstra’s Algorithm

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Initially \(S = \{s\}\) and \(d(s) = 0\)
While \(S \neq V\)
  - Select a node \(v \notin S\) with at least one edge from \(S\) for which
    \[
    d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e \text{ is as small as possible}
    \]
  - Add \(v\) to \(S\) and define \(d(v) = d'(v)\)
EndWhile

How many iterations are there of the while loop?
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Initially \(S = \{s\}\) and \(d(s) = 0\)

While \(S \neq V\)

Select a node \(v \notin S\) with at least one edge from \(S\) for which

\[d'(v) = \min_{e=(u,v),u \in S} d(u) + \ell_e\]

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EndWhile

How many iterations are there of the while loop? \(n - 1\).
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Let \(S\) be the set of explored nodes
For each \(u \in S\), we store a distance \(d(u)\)
Initially \(S = \{s\}\) and \(d(s) = 0\)
While \(S \neq V\)
    Select a node \(v \notin S\) with at least one edge from \(S\) for which
    \[
    d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e
    \]
    is as small as possible
    Add \(v\) to \(S\) and define \(d(v) = d'(v)\)
EndWhile

- How many iterations are there of the while loop? \(n - 1\).
- In each iteration, for each node \(v \notin S\), compute

\[
    d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e
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.
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   For each \(u \in S\), we store a distance \(d(u)\)
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   Select a node \(v \notin S\) with at least one edge from \(S\) for which
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   Add \(v\) to \(S\) and define \(d(v) = d'(v)\)
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- How many iterations are there of the while loop? \(n - 1\).
- In each iteration, for each node \(v \notin S\), compute
  \[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \]
- Running time per iteration is
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For each \(u \in S\), we store a distance \(d(u)\)

Initially \(S = \{s\}\) and \(d(s) = 0\)

While \(S \neq V\)

Select a node \(v \notin S\) with at least one edge from \(S\) for which

\[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \] is as small as possible

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

EndWhile

▶ How many iterations are there of the while loop? \(n - 1\).
▶ In each iteration, for each node \(v \notin S\), compute

\[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \]

. 

▶ Running time per iteration is \(O(m)\) ⇒ overall running time is \(O(nm)\).
A Faster implementation of Dijkstra’s Algorithm

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Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

EndWhile

- Observation: If we add \(v\) to \(S\), \(d'(w)\) changes only for \(v\)'s neighbours.
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   \[d'(v) = \min_{e=(u,v)\,u \in S} d(u) + \ell_e\]
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   Add \(v\) to \(S\) and define \(d(v) = d'(v)\)
EndWhile

- Observation: If we add \(v\) to \(S\), \(d'(w)\) changes only for \(v\)'s neighbours.
- Store the minima \(d'(v)\) for each node \(v \in V - S\) in a priority queue.
- Determine the next node \(v\) to add to \(S\) using \textsc{ExtractMin}.
- After adding \(v\) to \(S\), for each neighbour \(w\) of \(v\), compute \(d(v) + l_{(v,w)}\).
- If \(d(v) + l_{(v,w)} < d'(w)\),
   1. Set \(d'(w) = d(v) + l_{(v,w)}\).
   2. Update \(w\)'s key to the new value of \(d'(w)\) using \textsc{ChangeKey}.
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- How many times are \textsc{ExtractMin} and \textsc{ChangeKey} invoked?
A Faster implementation of Dijkstra’s Algorithm

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- Observation: If we add \(v\) to \(S\), \(d'(w)\) changes only for \(v\)’s neighbours.
- Store the minima \(d'(v)\) for each node \(v \in V - S\) in a priority queue.
- Determine the next node \(v\) to add to \(S\) using \texttt{EXTRACTMIN}.
- After adding \(v\) to \(S\), for each neighbour \(w\) of \(v\), compute \(d(v) + l_{(v,w)}\).
- If \(d(v) + l_{(v,w)} < d'(w)\),
  1. Set \(d'(w) = d(v) + l_{(v,w)}\).
  2. Update \(w\)’s key to the new value of \(d'(w)\) using \texttt{CHANGEKEY}.
- How many times are \texttt{EXTRACTMIN} and \texttt{CHANGEKEY} invoked? \(n - 1\) and \(m\) times, respectively.
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  2. Update \(w\)'s key to the new value of \(d'(w)\) using \texttt{ChangeKey}.
- How many times are \texttt{ExtractMin} and \texttt{ChangeKey} invoked? \(n - 1\) and \(m\) times, respectively. Total running time is \(O(m \log n)\).
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length.
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length. This problem is the NP-complete traveling salesman problem.
Minimum Spanning Tree (MST)

- Given an undirected graph \( G(V, E) \) with a cost \( c_e > 0 \) associated with each edge \( e \in E \).
- Find a subset \( T \) of edges such that the graph \( (V, T) \) is connected and the cost \( \sum_{e \in T} c_e \) is as small as possible.
Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
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**Minimum Spanning Tree**

**INSTANCE:** An undirected graph $G(V, E)$ and a function $c : E \to \mathbb{R}^+$

**SOLUTION:** A set $T \subseteq E$ of edges such that $(V, T)$ is connected and the $\sum_{e \in T} c_e$ is as small as possible.
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**Minimum Spanning Tree**

**INSTANCE:** An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

**SOLUTION:** A set $T \subseteq E$ of edges such that $(V, T)$ is connected and the $\sum_{e \in T} c_e$ is as small as possible.

- Claim: If $T$ is a minimum-cost solution to this network design problem then $(V, T)$ is a tree.
- A subset $T$ of $E$ is a *spanning tree* of $G$ if $(V, T)$ is a tree.
Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to \( T \) if tree property is not violated.
Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to $T$ if tree property is not violated.

**Increasing cost order**  Process edges in increasing order of cost. Discard an edge if it creates a cycle.

**Dijkstra-like**  Start from a node $s$ and grow $T$ outward from $s$: add the node that can be attached most cheaply to current tree.

**Decreasing cost order**  Delete edges in order of decreasing cost as long as graph remains connected.
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- Which of these algorithms works?
Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to $T$ if tree property is not violated.

  **Increasing cost order** Process edges in increasing order of cost. Discard an edge if it creates a cycle. *Kruskal’s algorithm*

  **Dijkstra-like** Start from a node $s$ and grow $T$ outward from $s$: add the node that can be attached most cheaply to current tree. *Prim’s algorithm*

  **Decreasing cost order** Delete edges in order of decreasing cost as long as graph remains connected. *Reverse-Delete algorithm*

- Which of these algorithms works? All of them!
Greedy Algorithm for the MST Problem

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**Decreasing cost order** Delete edges in order of decreasing cost as long as graph remains connected. **Reverse-Delete algorithm**

- Which of these algorithms works? All of them!
- Simplifying assumption: all edge costs are distinct.
Figure 4.9 Sample run of the Minimum Spanning Tree Algorithms of (a) Prim and (b) Kruskal, on the same input. The first 4 edges added to the spanning tree are indicated by solid lines; the next edge to be added is a dashed line.
Characterising MSTs

- Does the edge of smallest cost belong to an MST?
Characterising MSTs

- Does the edge of smallest cost belong to an MST? Yes.
- Which edges must belong to an MST?
Characterising MSTs

- Does the edge of smallest cost belong to an MST? Yes.
- Which edges must belong to an MST?
  - What happens when we delete an edge from an MST?
  - MST breaks up into sub-trees.
  - Which edge should we add to join them?
- Which edges cannot belong to an MST?
- What happens when we add an edge to an MST?
  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?
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- Which edges cannot belong to an MST?
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  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?
A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).

Every set $S \subset V$ ($S$ cannot be empty or the entire set $V$) has a corresponding cut: $\text{cut}(S)$ is the set of edges $(v, w)$ such that $v \in S$ and $w \in V - S$. 
Graph Cuts

- A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
- Every set $S \subseteq V$ ($S$ cannot be empty or the entire set $V$) has a corresponding cut: cut($S$) is the set of edges $(v, w)$ such that $v \in S$ and $w \in V - S$.
- cut($S$) is a cut because deleting the edges in cut($S$) disconnects $S$ from $V - S$. 
Cut Property

- When is it safe to include an edge in an MST?

Let $S \subset V$, $S$ is not empty or equal to $V$.

Let $e$ be the cheapest edge in cut($S$).

Claim: every MST contains $e$.

Proof: exchange argument. If a supposed MST $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$. 
Cut Property

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- Let $S \subset V$, $S$ is not empty or equal to $V$.
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- Let $e$ be the cheapest edge in $\text{cut}(S)$.
- Claim: every MST contains $e$.
- Proof: exchange argument. If a supposed MST $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

![Diagram showing the swapping of edge $e$ for edge $e'$ in the spanning tree $T$, as described in the proof of (4.17).]
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:
1. Start with an empty set $T$ of edges.
2. Process edges in $E$ in increasing order of cost.
3. Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

Claim: Kruskal’s algorithm outputs an MST.
Optimality of Kruskal’s Algorithm

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  1. Start with an empty set $T$ of edges.
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- Claim: Kruskal’s algorithm outputs an MST.
  1. For every edge $e$ added, demonstrate the existence of $S$ and $V - S$ such that $e$ and $S$ satisfy the cut property.
  2. Prove that the algorithm computes a spanning tree.
Optimality of Prim’s Algorithm

- Prim’s algorithm: Maintain a tree $(S, U)$
  1. Start with an arbitrary node $s \in S$ and $U = \emptyset$.
  2. Add the node $v$ to $S$ and the edge $e$ to $U$ that minimise

\[
\min_{e=(u,v),u\in S,v \notin S} c_e \equiv \min_{e \in \text{cut}(S)} c_e.
\]

3. Stop when $S = V$.

- Claim: Prim’s algorithm outputs an MST.
Optimality of Prim’s Algorithm

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1. Start with an arbitrary node \(s \in S\) and \(U = \emptyset\).
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\min_{e=(u,v), u \in S, v \notin S} c_e = \min_{e \in \text{cut}(S)} c_e.
\]

3. Stop when \(S = V\).

Claim: Prim’s algorithm outputs an MST.

1. Prove that every edge inserted satisfies the cut property.
2. Prove that the graph constructed is a spanning tree.
Cycle Property

- When can we be sure that an edge cannot be in any MST?

Let $C$ be any cycle in $G$ and let $e = (v, w)$ be the most expensive edge in $C$.

Claim: $e$ does not belong to any MST of $G$.

Proof: exchange argument. If a supposed MST $T$ contains $e$, show that there is a tree with smaller cost than $T$ that does not contain $e$. 

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CS 4104: Greedy Graph Algorithms
Cycle Property

- When can we be sure that an edge cannot be in any MST?
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Proof: exchange argument. If a supposed MST $T$ contains $e$, show that there is a tree with smaller cost than $T$ that does not contain $e$. 

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- When can we be sure that an edge cannot be in any MST?
- Let $C$ be any cycle in $G$ and let $e = (v, w)$ be the most expensive edge in $C$.
- Claim: $e$ does not belong to any MST of $G$.
- Proof: exchange argument. If a supposed MST $T$ contains $e$, show that there is a tree with smaller cost than $T$ that does not contain $e$.

![Figure 4.11 Swapping the edge $e'$ for the edge $e$ in the spanning tree $T$, as described in the proof of (4.20).](image)
Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set $E'$ of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
  - Stop after processing all the edges.

- Claim: the Reverse-Delete algorithm outputs an MST.
Optimality of the Reverse-Delete Algorithm

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  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
  - Stop after processing all the edges.

- Claim: the Reverse-Delete algorithm outputs an MST.
  1. Show that every edge deleted belongs to no MST.
  2. Prove that the graph remaining at the end is a spanning tree.
Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!
Implementing Prim’s Algorithm

- Maintain a tree \((S, U)\).
  1. Start with an arbitrary node \(s \in V\) and \(U = \emptyset\).
  2. Add the node \(v\) to \(S\) and the edge \(e\) to \(U\) that minimise
     \[
     \min_{e \in \text{cut}(S)} c_e.
     \]
  3. Stop when \(S = V\).
Implementing Prim’s Algorithm

- Maintain a tree \((S, U)\).
  1. Start with an arbitrary node \(s \in V\) and \(U = \emptyset\).
  2. Add the node \(v\) to \(S\) and the edge \(e\) to \(U\) that minimise \(\min_{e \in \text{cut}(S)} c_e\).
  3. Stop when \(S = V\).

- Sorting edges takes \(O(m \log n)\) time.
- Implementation is very similar to Dijkstra’s algorithm.
- Maintain \(S\) and store attachment costs \(a(v) = \min_{e \in \text{cut}(S)} c_e\) for every node \(v \in V - S\) in a priority queue.
- At each step, extract minimum \(v\) from priority queue and update the attachment costs of the neighbours of \(v\).
- Total of \(n - 1\) \textbf{ExtractMin} and \(m\) \textbf{ChangeKey} operations, yielding a running time of \(O(m \log n)\).
Implementing Kruskal’s Algorithm

1. Start with an empty set $T$ of edges.
2. Process edges in $E$ in increasing order of cost.
3. Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle.
Implementing Kruskal’s Algorithm

1. Start with an empty set \( T \) of edges.
2. Process edges in \( E \) in increasing order of cost.
3. Add the next edge \( e \) to \( T \) only if adding \( e \) does not create a cycle.

▶ Sorting edges takes \( O(m \log n) \) time.
▶ Key question in step 3: “Does adding \( e = (u, v) \) to \( T \) create a cycle?”
▶ Maintain set of connected components of \( T \) in a data structure that supports:
  ▶ \( \text{Find}(u) \): return the name of the connected component of \( T \) containing \( u \).
  ▶ \( \text{Union}(A, B) \): merge connected components \( A \) and \( B \).
▶ Implementing step 3: Adding \( e = (u, v) \) creates a cycle if and only if \( \text{Find}(u) = \text{Find}(v) \).
  3. If \( \text{Find}(u) \neq \text{Find}(v) \),
Implementing Kruskal’s Algorithm

1. Start with an empty set $T$ of edges.
2. Process edges in $E$ in increasing order of cost.
3. Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle.

- Sorting edges takes $O(m \log n)$ time.
- Key question in step 3: “Does adding $e = (u, v)$ to $T$ create a cycle?”
- Maintain set of connected components of $T$ in a data structure that supports:
  - FIND($u$): return the name of the connected component of $T$ containing $u$.
  - UNION($A, B$): merge connected components $A$ and $B$.

- Implementing step 3: Adding $e = (u, v)$ creates a cycle if and only if FIND($u$) = FIND($v$).
  3. If FIND($u$) $\neq$ FIND($v$), execute UNION(FIND($u$), FIND($v$)) and add $e$ to $T$. 
Analysing Kruskal’s Algorithm

- How many \texttt{FIND} invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many \texttt{FIND} invocations does Kruskal’s algorithm need? $2m$.
- How many \texttt{UNION} invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many `FIND` invocations does Kruskal’s algorithm need? $2m$.
- How many `UNION` invocations does Kruskal’s algorithm need? $n - 1$. 

Total running time of Kruskal’s algorithm is $O(m \log n)$. 

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Analysing Kruskal’s Algorithm

- How many $\text{FIND}$ invocations does Kruskal’s algorithm need? $2m$.
- How many $\text{UNION}$ invocations does Kruskal’s algorithm need? $n - 1$.
- We will show two implementations of $\text{UNION-FIND}$:
  - Each $\text{FIND}$ takes $O(1)$ time, $k$ invocations of $\text{UNION}$ take $O(k \log k)$ time in total.
  - Each $\text{FIND}$ takes $O(\log n)$ time and each invocation of $\text{UNION}$ takes $O(1)$ time.
Analysing Kruskal’s Algorithm

- How many \texttt{Find} invocations does Kruskal’s algorithm need? \(2m\).
- How many \texttt{Union} invocations does Kruskal’s algorithm need? \(n - 1\).
- We will show two implementations of \texttt{Union-Find}:
  - Each \texttt{Find} takes \(O(1)\) time, \(k\) invocations of \texttt{Union} take \(O(k \log k)\) time in total.
  - Each \texttt{Find} takes \(O(\log n)\) time and each invocation of \texttt{Union} takes \(O(1)\) time.
- Total running time of Kruskal’s algorithm is \(O(m \log n)\).
Abstraction of the data structure needed by Kruskal’s algorithm.

- Maintain disjoint subsets of elements from a universe $U$ of $n$ elements.
  - Think of each subset being a connected component of $T$.
- Each subset has a name. A subset’s name will be the identity of some element in it.
- Support three operations:
  1. $\text{MAKE-UNION-FIND}(U)$: initialise the data structure with elements in $U$.
  2. $\text{FIND}(u)$: return the identity of the subset that contains $u$.
  3. $\text{UNION}(A, B)$: merge the sets named $A$ and $B$ into one set.
Union-Find Data Structure: Implementation 1

- Running example: three sets \{s, u, w\}, \{t, v, z\}, \{i, j, x, y\} with names \(u\), \(v\), and \(j\), respectively.

- Store all the elements of \(U\) in an array Component.
- Assume identities of elements are integers from 1 to \(n\).
- Component\[s\] is the name of the set containing \(s\).
- Implementing the operations:
  1. MakeUnionFind(\(U\)): For each \(s \in U\), set Component\[s\] = \(s\) in \(O(n)\) time.
  2. Find(\(s\)): return Component\[s\] in \(O(1)\) time.
  3. Union(A, B): merge B into A by scanning Component and updating each index whose value is B to the value A. Takes \(O(n)\) time.

- Union is very slow because we cannot efficiently find the elements that belong to a given set.
Union-Find Data Structure: Implementation 1

- Running example: three sets \{s, u, w\}, \{t, v, z\}, \{i, j, x, y\} with names \textit{u}, \textit{v}, and \textit{j}, respectively.

- Store all the elements of \textit{U} in an array \texttt{COMPONENT}.
  - Assume identities of elements are integers from 1 to \textit{n}.
  - \texttt{COMPONENT}[s] is the name of the set containing \textit{s}.

- Implementing the operations:
Union-Find Data Structure: Implementation 1

- Running example: three sets \( \{s, u, w\}, \{t, v, z\}, \{i, j, x, y\} \) with names \( u, v, \) and \( j \), respectively.

- Store all the elements of \( U \) in an array \( \text{COMPONENT} \).
  - Assume identities of elements are integers from 1 to \( n \).
  - \( \text{COMPONENT}[s] \) is the name of the set containing \( s \).

- Implementing the operations:
  1. \( \text{MAKEUNIONFIND}(U) \): For each \( s \in U \), set \( \text{COMPONENT}[s] = s \) in \( O(n) \) time.
  2. \( \text{FIND}(s) \): return \( \text{COMPONENT}[s] \) in \( O(1) \) time.
  3. \( \text{UNION}(A, B) \): merge \( B \) into \( A \) by scanning \( \text{COMPONENT} \) and updating each index whose value is \( B \) to the value \( A \). Takes \( O(n) \) time.
Union-Find Data Structure: Implementation 1

- Running example: three sets \( \{s, u, w\}, \{t, v, z\}, \{i, j, x, y\} \) with names \( u, v, \) and \( j \), respectively.
- Store all the elements of \( U \) in an array \( \text{Component} \).
  - Assume identities of elements are integers from 1 to \( n \).
  - \( \text{Component}[s] \) is the name of the set containing \( s \).
- Implementing the operations:
  1. \text{MakeUnionFind} (\( U \)): For each \( s \in U \), set \( \text{Component}[s] = s \) in \( O(n) \) time.
  2. \text{Find} (\( s \)): return \( \text{Component}[s] \) in \( O(1) \) time.
  3. \text{Union} (\( A, B \)): merge \( B \) into \( A \) by scanning \( \text{Component} \) and updating each index whose value is \( B \) to the value \( A \). Takes \( O(n) \) time.
- \text{Union} is very slow because
Union-Find Data Structure: Implementation 1

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  1. \( \text{MAKEUNIONFIND}(U) \): For each \( s \in U \), set \( \text{COMPONENT}[s] = s \) in \( O(n) \) time.
  2. \( \text{FIND}(s) \): return \( \text{COMPONENT}[s] \) in \( O(1) \) time.
  3. \( \text{UNION}(A, B) \): merge \( B \) into \( A \) by scanning \( \text{COMPONENT} \) and updating each index whose value is \( B \) to the value \( A \). Takes \( O(n) \) time.

- \( \text{UNION} \) is very slow because we cannot efficiently find the elements that belong to a given set.
Optimisation 1: Use an array `Elements` in addition to `Component`.
- Indices of `Elements` range from 1 to $n$.
- `Elements[s]` stores the elements in the subset named $s$ in a list.

Execute `Union(A, B)` by merging $B$ into $A$ in two steps:
1. For every element $u \in B$, set $\text{Component}[u] = A$ in $O(|B|)$ time.
2. Append $\text{Elements}[B]$ to $\text{Elements}[A]$ in $O(1)$ time.

`Union` takes $\Omega(n)$ in the worst-case.
Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array \texttt{Elements} in addition to \texttt{Component}.
  - Indices of \texttt{Elements} range from 1 to \( n \).
  - \texttt{Elements}[s] stores the elements in the subset named \( s \) in a list.
- Execute \texttt{Union}(A, B) by merging \( B \) into \( A \) in two steps:
  1. For every element \( u \in B \), set \texttt{Component}[u] = A in \( O(|B|) \) time.
  2. Append \texttt{Elements}[B] to \texttt{Elements}[A] in \( O(1) \) time.
- \texttt{Union} takes \( \Omega(n) \) in the worst-case.
- Optimisation 2: Store size of each set in an array \texttt{Size}. If \( \texttt{Size}[B] \leq \texttt{Size}[A] \), merge \( B \) into \( A \). Otherwise merge \( A \) into \( B \). Update \texttt{Size}. 

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Union-Find Data Structure: Analysis of Implementation 2

- \texttt{MakeUnionFind}(S) and \texttt{Find}(u) are as before.

- \texttt{Union}(A, B): Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.

- Any sequence of $k$ \texttt{Union} operations takes $O(k \log k)$ time.

- $k$ \texttt{Union} operations touch at most $2k$ elements.

- Intuition: running time of \texttt{Union} is dominated by updates to \texttt{Component}.
  Charge each update to the element being updated and bound number of charges per element.

- Consider any element $s$. Every time $s$'s set identity is updated, the size of the set containing $s$ at least doubles $\Rightarrow s$'s set can change at most $\log(2k)$ times $\Rightarrow$ the total work done in $k$ \texttt{Union} operations is $O(k \log k)$.

- \texttt{Find} is fast in the worst case, \texttt{Union} is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: 
Analysis of Implementation 2

- MakeUnionFind($S$) and Find($u$) are as before.
- Union($A, B$): Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
Union-Find Data Structure: Analysis of Implementation 2

- \texttt{MakeUnionFind}(S) and \texttt{Find}(u) are as before.
- \texttt{Union}(A, B): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).
- Any sequence of \(k\) \texttt{Union} operations takes \(O(k \log k)\) time.
Union-Find Data Structure: Analysis of Implementation 2

- **MAKEUNIONFIND**\((S)\) and **FIND**\((u)\) are as before.

- **UNION**\((A, B)\): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).

- Any sequence of \(k\) **UNION** operations takes \(O(k \log k)\) time.
  - \(k\) **UNION** operations touch at most \(2k\) elements.

Find is fast in the worst case, **UNION** is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Analysis of Implementation 2

- **MAKEUNIONFIND**$(S)$ and **FIND**$(u)$ are as before.
- **UNION**$(A, B)$: Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ **UNION** operations takes $O(k \log k)$ time.
  - $k$ **UNION** operations touch at most $2k$ elements.
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Union-Find Data Structure: 
Analysis of Implementation 2

- **MakeUnionFind**(S) and **Find**(u) are as before.

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- Any sequence of $k$ **Union** operations takes $O(k \log k)$ time.
  - $k$ **Union** operations touch at most $2k$ elements.
  - Intuition: running time of **Union** is dominated by updates to **Component**. Charge each update to the element being updated and bound number of charges per element.
  - Consider any element $s$. Every time $s$'s set identity is updated, the size of the set containing $s$ at least doubles $\Rightarrow$ $s$'s set can change at most $\log(2k)$ times $\Rightarrow$ the total work done in $k$ **Union** operations is $O(k \log k)$.

- **Find** is fast in the worst case, **Union** is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Analysis of Implementation 2

- **MakeUnionFind**(S) and **Find**(u) are as before.
- **Union**(A, B): Running time is proportional to the size of the smaller set, which may be Ω(n).
- Any sequence of k **Union** operations takes \( O(k \log k) \) time.
  - k **Union** operations touch at most 2k elements.
  - Intuition: running time of **Union** is dominated by updates to **Component**. Charge each update to the element being updated and bound number of charges per element.
  - Consider any element s. Every time s’s set identity is updated, the size of the set containing s at least doubles ⇒ s’s set can change at most \( \log(2k) \) times ⇒ the total work done in k **Union** operations is \( O(k \log k) \).
- **Find** is fast in the worst case, **Union** is fast in an amortised sense. Can we make both operations worst-case efficient?
Goal: Implement FIND in $O(\log n)$ and UNION in $O(1)$ worst-case time.
Goal: Implement $\text{FIND}$ in $O(\log n)$ and $\text{UNION}$ in $O(1)$ worst-case time.

Represent each subset in a tree using pointers:
- Each tree node contains an element and a pointer to a parent.
- The identity of the set is the identity of the element at the root.

Figure 4.12 A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query $\text{Find}(i)$ would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Goal: Implement \textsc{Find} in $O(\log n)$ and \textsc{Union} in $O(1)$ worst-case time.

Represent each subset in a tree using pointers:
- Each tree node contains an element and a pointer to a parent.
- The identity of the set is the identity of the element at the root.

Implementing \textsc{Find}(u): follow pointers from $u$ to the root of $u$'s tree.

Implementing \textsc{Union}(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.

\textit{Figure 4.12} A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last \textsc{Union} operation. To answer a \textsc{Find} query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \textsc{Find}(i) would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Goal: Implement $\text{FIND}$ in $O(\log n)$ and $\text{UNION}$ in $O(1)$ worst-case time.

Represent each subset in a tree using pointers:

- Each tree node contains an element and a pointer to a parent.
- The identity of the set is the identity of the element at the root.

Implementing $\text{FIND}(u)$: follow pointers from $u$ to the root of $u$'s tree.

Implementing $\text{UNION}(A, B)$: make smaller tree’s root a child of the larger tree’s root. Takes $O(1)$ time.

![Diagram](image)

The set \{s, u, w\} was merged into \{t, v, z\}.

*Figure 4.12* A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query $\text{Find}(i)$ would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Why does $\text{Find}(u)$ take $O(\log n)$ time?

**Figure 4.12** A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query $\text{Find}(i)$ would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Why does $\text{Find}(u)$ take $O(\log n)$ time?

Number of pointers followed equals the number of times the identity of the set containing $u$ changed.

Every time $u$'s set's identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.
Every time we invoke \( \text{FIND}(u) \), we follow the same set of pointers.

Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes \( v \) and \( j \). The dashed arrow from \( u \) to \( v \) is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query Find(i) would involve following the arrows \( i \) to \( x \), and then \( x \) to \( j \).
Union-Find Data Structure: Improving Implementation 3

Every time we invoke FIND($u$), we follow the same set of pointers.

Path compression: make all nodes visited by FIND($u$) children of the root.

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Union-Find Data Structure: Improving Implementation 3

- Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.
- Path compression: make all nodes visited by $\text{FIND}(u)$ children of the root.

![Diagram showing path compression](image)

**Figure 4.13** (a) An instance of a Union-Find data structure; and (b) the result of the operation $\text{Find}(v)$ on this structure, using path compression.
Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.

Path compression: make all nodes visited by $\text{FIND}(u)$ children of the root.

Can prove that total time taken by $n$ $\text{FIND}$ operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows extremely slowly with $n$. 

Figure 4.13 (a) An instance of a Union-Find data structure; and (b) the result of the operation $\text{Find}(u)$ on this structure, using path compression.
Comments on Union-Find and MST

- The **Union-Find** data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.