Linear Algebra Examples and Summary

1: Subspaces

Basic Fact: A subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ is a nonempty subset $\mathcal{V}$ which is closed under addition and scalar multiplication.

Example 1 (a subspace of $\mathbf{R}^3$)

Let $\mathcal{W}$ be the set of all vectors of the form $(3a+2b, a, b)$. Show that $\mathcal{W}$ is a subspace of $\mathbf{R}^3$.

Note that a vector is in $\mathcal{W}$ if and only if its first component equals three times its second component plus two times its third component — that's a key point in the arguments below. We must show three things:

- $\mathcal{W}$ is nonempty:

The vector $\mathbf{0} = (0,0,0)$ is of the form $(3a+2b, a, b)$ — just let $a = b = 0$. So $\mathbf{0}$ is in $\mathcal{W}$.

- $\mathcal{W}$ is closed under scalar multiplication:

Suppose that $\mathbf{x}_1$ is in $\mathcal{W}$ and c is a scalar. Then $\mathbf{x}_1 = (3a_1 + 2b_1, a_1, b_1)$ for some real numbers a_1 and b_1 (by definition of $\mathcal{W}$). Thus

$$c\mathbf{x}_1 = (3ca_1 + 2cb_1, ca_1, cb_1),$$

which is of the right form to be in $\mathcal{W}$.

- $\mathcal{W}$ is closed under addition:

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $\mathcal{W}$. Then $\mathbf{x}_1 = (3a_1 + 2b_1, a_1, b_1)$ and $\mathbf{x}_2 = (3a_2 + 2b_2, a_2, b_2)$ for some real numbers a_1, b_1, a_2, and b_2 (by definition of $\mathcal{W}$). Thus

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= (3a_1 + 2b_1, a_1, b_1) + (3a_2 + 2b_2, a_2, b_2) \\ &= (3(a_1 + a_2) + 2(b_1 + b_2), a_1 + a_2, b_1 + b_2), \end{aligned}$$

which is of the right form to be in $\mathcal{W}$. Therefore, $\mathcal{W}$ is a subspace of $\mathbf{R}^3$.

$\blacksquare$

Example 2 (not a subspace)

Let $\mathcal{H} = \{(a, b, a+1) \ : \ a, b \in \mathbf{R}\}$. Is $\mathcal{H}$ a subspace of $\mathbf{R}^3$?

Consider the same three properties as in example 1.

- Is $\mathcal{H}$ nonempty?

Yes — for example, $(1, 1, 2) \in \mathcal{H}$. However, the zero vector $(0, 0, 0)$ is not in $\mathcal{H}$, and that is enough to tell us that $\mathcal{H}$ is not a subspace.

- Is $\mathcal{H}$ closed under scalar multiplication?

No — a vector is in $\mathcal{H}$ if and only if its third component equals its first component plus 1. Recall that the vector $(1, 1, 2) \in \mathcal{H}$. But if you multiply that vector by any scalar (other than 1), the result is not in $\mathcal{H}$. For instance, $2(1, 1, 2) = (2, 2, 4)$, which is not of the right form.

- Is $\mathcal{H}$ closed under addition?

Again, no — if you add two vectors that are in $\mathcal{H}$, the result is not in $\mathcal{H}$. For instance, $(1, 0, 2)$ and $(0, 1, 1)$ are in $\mathcal{H}$, but their sum $(1, 1, 3)$ is not.

Be sure you're clear on the logic of all this. To show that $\mathcal{H}$ is not a subspace, you only have to show that one of the properties does not hold. We considered all three just for entertainment. Also, be sure you understand the idea of a <u>counterexample</u>. To show that $\mathcal{H}$ <u>is not</u> closed under addition, all we need to do is find one case where the sum of two vectors in $\mathcal{H}$ is not in $\mathcal{H}$. But if we want to show that $\mathcal{H}$ <u>is</u> closed under addition, we must show that closure holds no matter which vectors from $\mathcal{H}$ we pick — so we must make a <u>general</u> argument as in example 1.

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<u>Example 3 (a polynomial subspace)</u>

Let $\mathcal{V} = \{ax^2 \ : \ a \in \mathbf{R}\}$. Is $\langle$ a subspace of P_2, the vector space of all polynomials of degree 2 or less?

Consider the same three properties as in example 1.

- Is $\mathcal{V}$ nonempty?

Obviously, for example, if we let $a = 5$, we see that the polynomial $5x^2 \in \mathcal{V}$. Also, the zero polynomial 0 is in $\mathcal{V}$.

- Is $\mathcal{V}$ closed under scalar multiplication?

If we take two polynomials in $\mathcal{V}$, say ax^2 and bx^2, where a and b are scalars, their sum is the polynomial $(a + b)x^2$, which is in $\mathcal{V}$, since it's a scalar times x^2.

- Is $\mathcal{V}$ closed under addition?

If we take a polynomial ax^2 in $\mathcal{V}$, and a scalar α, then $\alpha(ax^2) = (\alpha a)x^2$, which is in $\mathcal{V}$.

So $\mathcal{V}$ is a subspace of P_2.

$\blacksquare$

2: Linear Combinations

Basic Problem: Given a vector $\mathbf{w}$ and a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, are there scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n?$$

If so, then we say that $\mathbf{w}$ is a linear combination of the vectors in S.

Procedure: Let A be the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The scalars c_k would be the solution of the linear system $A\mathbf{c} = \mathbf{w}$. So just reduce the augmented matrix $[A|\mathbf{w}]$ and see if there is a solution.

Example 4 (in $\mathbf{R}^4$):

Is the vector $\mathbf{w} = (3, 7, 8, 19)$ a linear combination of the vectors:

$$S = \{(-2, 3, 4, 1), (3, 2, 0, 5), (0, 3, 1, 0), (-2, 1, 4, 2), (1, 0, -1, 4)\}?$$

It's easy to verify that

$$\left[\begin{array}{ccccc|c} -2 & 3 & 0 & -2 & 1 & 3 \\ 3 & 2 & 3 & 1 & 0 & 7 \\ 4 & 0 & 1 & 4 & -1 & 8 \\ 1 & 5 & 0 & 2 & 4 & 19 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 35 & 34 \\ 0 & 1 & 0 & 0 & 5 & 7 \\ 0 & 0 & 1 & 0 & -29 & -28 \\ 0 & 0 & 0 & 1 & -28 & -25 \end{array}\right],$$

and so we have the general solution:

$$\begin{aligned} c_1 &= 34 - 35t \\ c_2 &= 7 - 5t \\ c_3 &= -28 + 29t \\ c_4 &= -25 + 28t \\ c_5 &= t \end{aligned}$$

Letting $c_5 = 1$ (for example) yields the particular solution $c_1 = -1$, $c_2 = 2$, $c_3 = 1$, $c_4 = 3$, and $c_5 = 1$. Thus we can write $\mathbf{w}$ as a linear combination of the vectors in S:

$$\mathbf{w} = -1(-2, 3, 4, 1) + 2(3, 2, 0, 5) + 1(0, 3, 1, 0) + 3(-2, 1, 4, 2) + 1(1, 0, -1, 4).$$

$\blacksquare$

<u>Example 5 (in a polynomial space)</u>:

Let P_2 be the set of all polynomials of degree two or less; then P_2 is a vector space. Is the vector $q(x) = 6 + 8x - 3x^2$ a linear combination of the vectors (polynomials)

$$B = \left\{x + 1, 2x - 1, 3x^2\right\}?$$

The answer is yes if, and only if, there are scalars c_1, c_2 and c_3 such that

$$c_1(x + 1) + c_2(2x - 1) + c_3(3x^2) = q(x),$$

which is the same as

$$(c_1 - c_2) + (c_1 + 2c_2)x + 3c_3x^2 = 6 + 8x - 3x^2$$

Equating the corresponding coefficients yields a linear system, whose augmented matrix reduces as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 1 & 2 & 0 & 8 \\ 0 & 0 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right],$$

which implies the solution $c_1 = 4$, $c_2 = 2$, and $c_3 = -1$. Thus

$$4(x + 1) + 2(2x - 1) - 1(3x^2) = q(x),$$

and so $q(x)$ is a linear combination of the vectors in B.

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<u>Example 6 (not a linear combination)</u>:

Is the vector $\mathbf{x} = (2, 1, 7)$ a linear combination of the vectors $\mathbf{v}_1 = (1, 0, -2)$ and $\mathbf{v}_2 = (0, 1, 3)$?

Using the procedure from example 4, we have

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -2 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \end{array}\right],$$

and so (by the third row) we see that there is no solution. Therefore, $\mathbf{x}$ is <u>not</u> a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

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3: Spanning

Basic Problem: Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, taken from a vector space $\mathcal{W}$, does S span $\mathcal{W}$? That is, for every vector $\mathbf{w}$ in $\mathcal{W}$, are there scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{w}\,?$$

Procedure: Let A be the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The scalars c_k would be the solution of the linear system $A\mathbf{c} = \mathbf{w}$. So just reduce the augmented matrix $[A|\mathbf{w}]$ and see if there is a nontrivial solution. The only pitfall is that $\mathbf{w}$ must be a generic description of the vectors in $\mathcal{W}$ — not a specific vector.

Example 7

Does $S = \{(5, 1, 1), (1, 1, -1)\}$ span the subspace $\mathcal{W}$ from example 1?

The answer depends on whether, for every vector $\mathbf{x}$ in $\mathcal{W}$ there are scalars c_1 and c_2 such that

$$\mathbf{x} = c_1(5, 1, 1) + c_2(1, 1, -1).$$

Now recall that $\mathbf{x}$ must be of the form $(3a + 2b, a, b)$, so if we just use the usual procedure for deciding whether $\mathbf{x}$ is a linear combination of a set of vectors, we discover that:

$$\left[\begin{array}{cc|c} 5 & 1 & 3a+2b \\ 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & (1/2)a + (1/2)b \\ 0 & 1 & (1/2)a - (1/2)b \\ 0 & 0 & 0 \end{array}\right]$$

So there is a solution for every choice of a and b (that is, for every possible choice of $\mathbf{x}$). Therefore, S does span the subspace $\mathcal{W}$.

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Example 8

Let P_2 be the set of all polynomials of degree two or less and again consider the set of vectors

$$B = \{x + 1, 2x - 1, 3x^2\}.$$

Does B span P_2? B spans P_2 if, and only if, for every vector $q(x)$ in P_2 there are scalars c_1, c_2 and c_3 such that

$$c_1(x + 1) + c_2(2x - 1) + c_3(3x^2) = q(x).$$

Now if $q(x)$ can be any element of P_2 all we can say about it is that $q(x) = a + bx + cx^2$ for some scalars a, b and c. Following the same procedure as before, we get a linear system which reduces as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 2 & 0 & b \\ 0 & 0 & 3 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a-b \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & c/3 \end{array}\right].$$

Thus, there <u>is</u> a solution for every choice of a, b and c and so B <u>does</u> span P_2. In particular, we can say that if $q(x) = a + bx + cx^2$ then

$$q(x) = (2a - b)(x + 1) + (b - a)(2x - 1) + (c/3)(3x^2).$$

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<u>**Example 9**</u>

Does $S = \{(2, 1, 4), (0, 1, 1), (2, -1, 2)\}$ span $\mathbf{R}^3$?

First note that if $\mathbf{x}$ is in $\mathbf{R}^3$ all we can really say about it is that $\mathbf{x} = (a, b, c)$ for some real numbers a, b and c. Following exactly the same logic as in example 7, we discover that:

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & a \\ 1 & 1 & -1 & b \\ 4 & 1 & 2 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -a - b + c \\ 0 & 1 & -2 & a + 2b - c \\ 0 & 0 & 0 & 3a + 2b - 2c \end{array}\right]$$

Now that shows that there <u>won't always</u> be a solution. If a, b and c are chosen so that $3a + 2b - 2c = 0$ then there will — otherwise there won't. In other words, some vectors in $\mathbf{R}^3$ are linear combinations of the vectors in S and some are not. So S does not span $\mathbf{R}^3$. (But S <u>does</u> span some subspace of $\mathbf{R}^3$ — the plane $3x + 2y - 2z = 0$.)

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<u>**Example 10**</u>

Does $T = \{(2, 1, 4), (0, 1, 1), (2, -1, 2), (1, 0, 1)\}$ span $\mathbf{R}^3$?

Following exactly the same logic as in example 7, we discover that:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 1 & a \\ 1 & 1 & -1 & 0 & b \\ 4 & 1 & 2 & 1 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -a - b + c \\ 0 & 1 & -2 & 0 & a + 2b - c \\ 0 & 0 & 0 & 1 & 3a + 2b - 2c \end{array}\right]$$

Now that shows that there <u>will always</u> be a solution. In other words, every vector in $\mathbf{R}^3$ is a linear combination of the vectors in T. So S does span $\mathbf{R}^3$.

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4: Linear Independence

<u>Basic Problem</u>: Given a vector a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, is S linearly dependent or linearly independent? That is, are there scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

and the scalars are not all zero?

Procedure: Let A be the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The scalars c_k would be the solution of the homogeneous linear system $A\mathbf{c} = \mathbf{0}$. So just reduce the augmented matrix $[A|\mathbf{0}]$ and see if there is a nontrivial solution.

Example 11 (in $\mathbf{R}^4$):

Is the set of vectors:

$$S = \{(-2,3,4,1),(3,2,0,5),(0,3,1,0),(-2,1,4,2),(1,0,-1,4)\}\ ?$$

linearly dependent?

It's easy to verify that

$$\left[\begin{array}{rrrrr|r} -2 & 3 & 0 & -2 & 1 & 0 \\ 3 & 2 & 3 & 1 & 0 & 0 \\ 4 & 0 & 1 & 4 & -1 & 0 \\ 1 & 5 & 0 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 35 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -29 & 0 \\ 0 & 0 & 0 & 1 & -28 & 0 \end{array}\right],$$

Since the fifth variable is a parameter, the homogeneous linear system has nontrivial solutions. Therefore, S is linearly dependent. To see how to write the zero vector $\mathbf{0}$ as a nontrivial linear combination of the vectors in S you only need to find a particular (nontrivial) solution. From the reduced matrix, the general solution is

$$\begin{aligned} c_1 &= -35t \\ c_2 &= -5t \\ c_3 &= 29t \\ c_4 &= 28t \\ c_5 &= t \end{aligned}$$

and letting $c_5 = 1$ (for instance) yields the nontrivial solution $c_1 = -35$, $c_2 = -5$, $c_3 = 29$, $c_4 = 28$, and $c_5 = 1$. Thus we can write $\mathbf{0}$ as:

$$\mathbf{0} = -35(-2,3,4,1) - 5(3,2,0,5) + 29(0,3,1,0) + 28(-2,1,4,2) + 1(1,0,-1,4).$$

We can also write one of the vectors in S as a linear combination of the other four. For example, you can rearrange the equation above to get:

$$(-2,3,4,1) = \frac{-5}{35}(3,2,0,5) + \frac{29}{35}(0,3,1,0) + \frac{28}{35}(-2,1,4,2) + \frac{1}{35}(1,0,-1,4),$$

so the vector $\mathbf{v}_1$ depends on the other four vectors in S. (There's nothing special about choosing $\mathbf{v}_1$ here — you can do the same thing for any of the five vectors in S.)

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Example 12:

Recall that from example 10,

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 1 & a \\ 1 & 1 & -1 & 0 & b \\ 4 & 1 & 2 & 1 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -a-b+c \\ 0 & 1 & -2 & 0 & a+2b-c \\ 0 & 0 & 0 & 1 & 3a+2b-2c \end{array}\right].$$

and so obviously,

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 4 & 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

This shows that the vectors

$$\{(2,1,4),(0,1,1),(2,-1,2),(1,0,1)\}$$

are linearly dependent.

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Example 13 (in $\mathbf{R}^n$):

Is the set of vectors:

$$T = \{(-2,3,4,1),(3,2,0,5),(0,3,1,0),(-2,1,4,2)\}\ ?$$

linearly dependent?

It's easy to verify that

$$\left[\begin{array}{cccc|c} -2 & 3 & 0 & -2 & 0 \\ 3 & 2 & 3 & 1 & 0 \\ 4 & 0 & 1 & 4 & 0 \\ 1 & 5 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

Since there are no parameters, this homogeneous linear system has only the trivial solution — therefore, T is linearly independent.

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Example 14 (in a polynomial space):

Let P_2 be the set of all polynomials of degree two or less and again consider the set of vectors

$$B = \left\{x+1, 2x-1, 3x^2\right\}.$$

Is B linearly independent or linearly dependent. B is linearly independent if, and only if, the only scalars c_1, c_2 and c_3 such that

$$c_1(x+1) + c_2(2x-1) + c_3(3x^2) = 0,$$

are $c_1 = c_2 = c_3 = 0$. Following the same procedure as before, we get a linear system which reduces as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

which implies the <u>only</u> solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Therefore, B <u>is</u> linearly independent.

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5: Basis and Dimension

<u>Basic Fact</u>: A set S of vectors from a vector space $\mathcal{V}$ is a <u>basis</u> for $\mathcal{V}$ if and only if the set S is linearly independent and spans $\mathcal{V}$. Every basis for a given vector space contains the same number of vectors, called the <u>dimension</u> of the vector space.

<u>Example 15</u>

Find a subset of T (from example 10) that is a <u>basis</u> for $\mathbf{R}^3$.

We need to find a linearly independent subset of T that spans $\mathbf{R}^3$. But that's easy if you look at the reduction given in example 10 in the right way. The first, second and fourth vectors in T become distinct (i.e. different) columns of the identity matrix when you reduce $[A|\mathbf{0}]$. And throwing out the third vector wouldn't change the operations you'd need to perform to reduce the matrix. So even without doing any work it's obvious that

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 4 & 1 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -a-b+c \\ 0 & 1 & 0 & a+2b-c \\ 0 & 0 & 1 & 3a+2b-2c \end{array}\right],$$

so the three vectors we used here span $\mathbf{R}^3$ and they are linearly independent since it's also obvious that

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore, $\{(2,1,4),(0,1,1),(1,0,1)\}$ is a basis for $\mathbf{R}^3$.

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Example 16

Let P_2 be the set of all polynomials of degree two or less and again consider the set of vectors

$$B = \left\{x+1, 2x-1, 3x^2\right\}.$$

Is B a basis for P_2? We know from example 14 that B is linearly independent, and from example 8 that B spans P_2. Therefore, B is a basis for P_2, and P_2 is a 3-dimensional vector space.

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6: The Kernel of a Matrix

Basic Fact: Given any $m \times n$ matrix A, the set of all solutions of the associated homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a vector space, called the kernel of the matrix A. The dimension of the kernel is called the nullity of A.

Example 17

Find a basis for the kernel of A and state the nullity, if A is a 3×4 matrix such that $[A|\mathbf{0}]$ reduces to:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is:

$$\mathbf{x} = (2a-b, a, b, b),$$

and a little algebra reveals that

$$\mathbf{x} = a(2,1,0,0) + b(-1,0,1,1).$$

The vectors $\mathbf{v}_1 = (2,1,0,0)$ and $\mathbf{v}_2 = (-1,0,1,1)$ are a basis for the solution space. Why? Well, the equation above shows that they span the solution space. And it's clear that they're linearly independent (look at how the zeros in one vector match up with nonzeros in the other one). So we know that the solution space of this homogeneous system has dimension 2.

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