Finding a minimum of a function. Example.

Problem: A Newton's method is used to find the minimum x^m of $f(x) = \cos(x-1) + \frac{1}{2}(x-1)^2$ for $0.95 < x < 1.05$, on a computer with $\epsilon_m = 10^{-8}$. Assuming that the initial guess x_0 is chosen wisely and the method converges normally, how many iterations N will it make before stopping? Assume that the stopping criteria are set correctly.

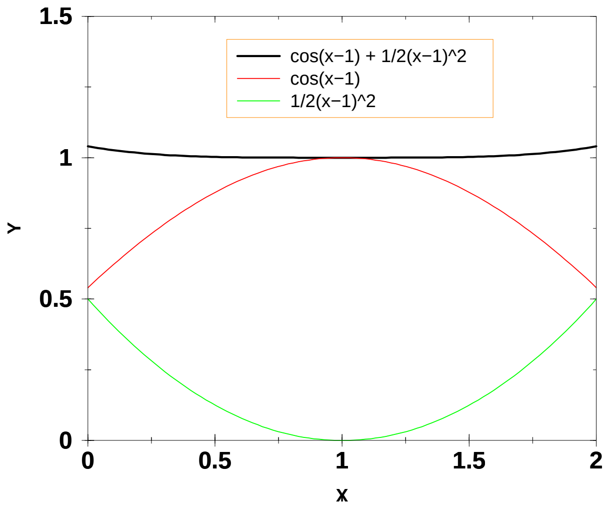

Figure 1: Function to minimize.

Solution: First, you need to realize that $N = N(|x_0 - x^m|, TOL, Method)$. The method is Newton's, so convergence is quadratic (see later). For $|x_0 - x^m|$ one can take the entire initial bracketing interval (unless initial x_0 is given, which is not the case in this example), $|x_0 - x^m| = 1.05 - 0.95 = 0.1$. The *generic* value for TOL (in the context of *minimization*) is $TOL = \sqrt{\epsilon_m} \times |x^m| = 10^{-4}$ *unless* there is something special about the function and we can do better (also note that obviously $x^m = 1$ in our specific case; in general, unless otherwise specified you can assume $x^m \sim 1$). So, let's stick to the above value of $TOL = 10^{-4}$ for now.

Let's see how the actual iterations might look like. Quadratic convergence means that $|x_n - x^m| \leq C \times |x_{n-1} - x^m|^2$. Unless otherwise specified, assume $C \sim 1$ and use the $=$ sign. So, for $n = 0$ we just take $|x_0 - x^m| = 0.1$ – our initial guess x_0 must be someplace within the $[0.95, 1.05]$ interval of length 0.1. For $n = 1$, $|x_1 - x^m| \sim |x_0 - x^m|^2 = 10^{-2}$, for $n = 2$, $|x_2 - x^m| \sim |x_1 - x^m|^2 = (10^{-2})^2 = 10^{-4}$, etc.

n	0	1	2	3	4	5
$\lvert x_n - x^m \rvert$	0.1	10^{-2}	10^{-4}	10^{-8}	10^{-16}	10^{-32}

Therefore, the TOL is reached at $n = 2$ (second iteration). You can stop here, but one can do a bit better. A sharp eye will notice that the function $f(x)$ is extremely flat around its minimum. This is because

$$f(x) \approx \underbrace{1 - \frac{1}{2}(x-1)^2 + \frac{1}{4!}(x-1)^4 + \ldots}_{Taylor\,for\,\cos(x-1)} + \tfrac{1}{2}(x-1)^2 \approx 1 + \tfrac{1}{24}(x-1)^4,$$

and so the closest you can get to the minimum is actually $TOL \sim {\epsilon_m}^{1/4} = 10^{-2}$ which gives $n = 1$ (first iteration). Of course, I won't expect you to be so discerning on an actual exam, I will give thick hints that $TOL \sim \sqrt{\epsilon_m}$ is not good enough.