B. Methods for solving a single IVP

1. Taylor series methods
   - Idea: approximate $x(t_{k+1})$ a truncated Taylor series for x, expanded about t_k.
   - Examples
     - Euler's method:
       $$x_{k+1} = x_k + hf(t_k, x_k)$$
     - Higher order methods
       $$x_{k+1} = x_k + hf(t_k, x_k) + \frac{h^2}{2} f'(t_k, x_k) + \cdots + \frac{h^p}{p!} f^{(p-1)}(t_k, x_k)$$

2. Runge-Kutta methods
   - Idea: get better approximation to $x(t+h)$ by evaluating f at more places, rather than by differentiating f.
   - Examples
     - RK method of order 2
       $$x_{k+1} = x_k + \frac{h}{2}\left[f(t_k, x_k) + f(t_{k+1}, x_k + hf(t_k, x_k))\right]$$
     - RK method of order 4
       $$\begin{aligned} K_1 &= hf(t_k, x_k), \\ K_2 &= hf\left(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_1\right), \\ K_3 &= hf\left(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_2\right), \\ K_4 &= hf(t_{k+1}, x_k + K_3), \\ x_{k+1} &= x_k + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned}$$

3. Multistep methods

   - Idea: use information from $t_0, \ldots, t_k$ to compute x_{k+1}.
   - Examples
     - An "explicit" m-step formula:

$$x_{k+1} = \sum_{j=k+1-m}^{k} \alpha_j x_j + \sum_{j=k+1-m}^{k} \beta_j f(t_j, x_j),$$

       where the α_j and β_j are chosen to minimize the local error.
     - An Adams-Bashforth m-step formula:

$$x_{k+1} = x_k + h \left(\sum_{j=k+1-m}^{k} \beta_j f(t_j, x_j) \right).$$

     - A 3-step Adams-Bashforth formula:

$$x_{k+1} = x_k + \frac{h}{12} \left(23 f(t_k, x_k) - 16 f(t_{k-1}, x_{k-1}) + 5 f(t_{k-2}, x_{k-2}) \right).$$

4. Implicit methods

   - Idea: x_{k+1} is defined implicitly at each step
   - Examples:
     - Backward Euler.

$$x_{k+1} = x_k + h f(t_{k+1}, x_{k+1})$$

     - Trapezoidal rule.

$$x_{k+1} = x_k + h(f(t_k, x_k) + f(t_{k+1}, x_{k+1}))$$

     - 3-step Adams-Moulton.

$$x_{k+1} = x_k + \frac{h}{24}(9 f(t_{k+1}, x_{k+1}) + 19 f(t_k, x_k) - 5 f(t_{i-1}, x_{i-1}) + f(t_{i-2}, x_{i-2}))$$