Using finite differences to discretize a BVP.

1. Idea: replace a continuous problem involving derivatives with a discrete problem involving finite differences.

2. Example. Find $x(t)$ satisfying

$$\begin{aligned} x''(t) &= f(t) \quad \text{on } [0,1] \\ x(0) &= 1 \\ x(1) &= 2 \end{aligned}$$

- Discretize the interval: let $t_i = i/n$, for $i = 0, \ldots, n$ and try to approximate $x_i \approx x(t_i)$.

- Recall that
$$x''(t) = \frac{x(t-h) - 2x(t) + x(t+h)}{h^2} + O(h^2).$$

- So a discrete approximation to the continuous problem would be: find $x_1, \ldots, x_{n-1}$ satisfying

$$\begin{aligned} x_{i-1} - 2x_i + x_{i+1} &= h^2 f_i \quad \text{for } i = 1, \ldots, n-1 \\ x_0 &= 1 \\ x_n &= 2 \end{aligned}$$

- Matrix formulation.

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 f_1 - 1 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{n-2} \\ h^2 f_{n-1} - 2 \end{pmatrix}$$