B. Direct methods: Gaussian elimination

1. Basic idea. Reduce $Ax = b$ to an *upper triangular* system by applying a series of elementary row operations:
   - Multiply an equation by a constant (*scaling*).
   - Interchange two equations (*pivoting*).
   - Add a multiple of one equation to another.

   So in the simplest case, we would ...

   $$\begin{array}{c} \text{Use eqn 1 to eliminate } x_1 \text{ from eqns } 2, \ldots, n. \\ \text{Use eqn 2 to eliminate } x_2 \text{ from eqns } 3, \ldots, n. \\ \vdots \\ \text{Use eqn } n-1 \text{ to eliminate } x_{n-1} \text{ from eqn } n. \end{array}$$

2. Example ...

3. The algorithm.

   - *Elimination* (to reduce $Ax = b$ to upper triangular form).

     for $k = 1$ to $n - 1$
       for $i = k + 1$ to n
         $m_{ik} = a_{ik}/a_{kk}$
         $b_i = b_i - m_{ik} b_k$
         for $j = k + 1$ to n
           $a_{ij} = a_{ij} - m_{ik} a_{kj}$

   - *Back substitution* (to solve upper triangular system $Ax = b$).

     $x_n = b_n/a_{nn}$
     for $j = n - 1$ downto 1
       $x_j = (b_j - \sum_{i=j+1}^{n} a_{ji} x_i)/a_{jj}$

   - Computational complexity.
     - Examples ...
     - Of back substitution.
     - Of elimination.

4. Pivoting.
   - Refers to row (and/or column) interchanging during elimination; we refer to a_{kk} is the "pivot."
   - Purpose:
     - Avoid division by zero. If pivot is 0, look down column for a nonzero, and interchange rows. (What if you don't find one?!)
     - Improve stability of Gaussian elimination.
   - "Partial" pivoting: search down column for element with largest absolute value, and switch rows.
   - "Complete" pivoting: search entire lower right submatrix for element with largest absolute value; switch rows and columns to make it the new pivot.
   - Remarks:
     - Complete pivoting is rarely used — too expensive.
     - Partial pivoting is usually good enough.
   - An example of the danger of small pivots ...

5. Matrix representation of Gaussian elimination.
   - Elimination algorithm (without pivoting) is equivalent to the *decomposition* (or *factorization*) $A = LU$, where

$$\begin{aligned} U &= \text{upper triangular matrix resulting from elimination} \\ L &= \text{unit lower triangular matrix with } l_{ik} = m_{ik}, i > k. \end{aligned}$$

   - With partial pivoting, elimination is equivalent to $PA = LU$, where P is a *permutation matrix.*
   - To solve $Ax = b$:

1)	factor $PA = LU$	(elimination)
2)	solve $Ly = Pb$	(forward substitution)
3)	solve $Ux = y$	(back substitution)

   - Important note: can repeat steps 2 and 3 for additional right hand sides b.