B. Direct methods: Gaussian elimination

1. Basic idea. Reduce $Ax = b$ to an upper triangular system by applying a series of elementary row operations:
   - Multiply an equation by a constant (*scaling*).
   - Interchange two equations (*pivoting*).
   - Add a multiple of one equation to another.

So in the simplest case, we would . . .

Use eqn 1 to eliminate $x_1$ from eqns 2, . . . , $n$.
Use eqn 2 to eliminate $x_2$ from eqns 3, . . . , $n$.

: 
Use eqn $n - 1$ to eliminate $x_{n-1}$ from eqn $n$.

2. Example . . .

3. The algorithm.

   - *Elimination* (to reduce $Ax = b$ to upper triangular form).
     
     for $k = 1$ to $n - 1$
     for $i = k + 1$ to $n$
     
     \[ m_{ik} = a_{ik}/a_{kk} \]
     \[ b_j = b_i - m_{ik}b_k \]
     for $j = k + 1$ to $n$
     
     \[ a_{ij} = a_{ij} - m_{ik}a_{kj} \]

   - *Back substitution* (to solve upper triangular system $Ax = b$).
     
     \[ x_n = b_n/a_{nn} \]
     for $j = n - 1$ downto 1
     \[ x_j = (b_j - \sum_{i=j+1}^{n} a_{ji}x_i)/a_{jj} \]

   - Computational complexity.
     
     - Examples . . .
     
     - Of back substitution.

     - Of elimination.
4. Pivoting.

- Refers to row (and/or column) interchanging during elimination; we refer to \( a_{kk} \) is the “pivot.”

- Purpose:
  - Avoid division by zero. If pivot is 0, look down column for a nonzero, and interchange rows. (What if you don’t find one?!)
  - Improve stability of Gaussian elimination.

- “Partial” pivoting: search down column for element with largest absolute value, and switch rows.

- “Complete” pivoting: search entire lower right submatrix for element with largest absolute value; switch rows and columns to make it the new pivot.

- Remarks:
  - Complete pivoting is rarely used — too expensive.
  - Partial pivoting is usually good enough.

- An example of the danger of small pivots . . .

5. Matrix representation of Gaussian elimination.

- Elimination algorithm (without pivoting) is equivalent to the decomposition (or factorization) \( A = LU \), where
  
  \[ U = \text{upper triangular matrix resulting from elimination} \]
  \[ L = \text{unit lower triangular matrix with } l_{ik} = m_{ik}, i > k. \]

- With partial pivoting, elimination is equivalent to \( PA = LU \), where \( P \) is a permutation matrix.

- To solve \( Ax = b \):
  1) factor \( PA = LU \) (elimination)
  2) solve \( Ly = Pb \) (forward substitution)
  3) solve \( Ux = y \) (back substitution)

- Important note: can repeat steps 2 and 3 for additional right hand sides \( b \).