Linear Algebra, Norms and Inner Products

I. Preliminaries

A. Definition: a *vector space* (*linear space*) consists of:

1. a field F of scalars. (We are interested in $F = \Re$).
2. a set V of vectors.
3. an operation $+$, called vector addition, which for all $x, y, z \in V$ satisfies:
   - $x + y \in V$
   - $x + y = y + x$
   - $x + (y + z) = (x + y) + z$
   - $\exists$ a zero vector 0, such that $x + 0 = x$
   - $\exists$ an inverse $-x$, such that $x + (-x) = 0$
4. an operation $\cdot$, called scalar multiplication, which for all $x, y \in V$ and $\alpha, \beta \in F$ satisfies:
   - $\alpha x \in V$
   - $\exists$ and identity 1, such that $1x = x$
   - $(\alpha\beta)x = \alpha(\beta x)$
   - $\alpha(x + y) = \alpha x + \alpha y$
   - $(\alpha + \beta)x = \alpha x + \beta x$

B. Examples

1. $V = \Re^n = \{x \,|\, x = (x_1, \ldots, x_n), x_i \in \Re\}$
   $x + y = (x_1 + y_1, \ldots, x_n + y_n)$
   $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$
2. $V = \Re^{m \times n} = \{m \times n \text{ matrices}\}$
   $(A + B)_{ij} = a_{ij} + b_{ij}$
   $(\alpha A)_{ij} = \alpha a_{ij}$
3. $V = P_n = \{\text{polynomials of degree } \le n\}$
4. $V = \{\text{continuous functions on } [0, 1]\}$

C. Definition: a *linear combination* of the vectors $x_1, \ldots, x_n$ is given by

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where $\alpha_1, \ldots, \alpha_n$ are scalars.

D. Definition: a set of vectors $\{x_1, \ldots, x_n\}$ is *linearly independent* iff

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \quad \Longrightarrow \quad \alpha_i = 0$$

E. Basis

- Definition: a linearly independent set of vectors $\{x_1, \ldots, x_n\}$ is a *basis* for a vector space V iff for every $x \in V$, there exist scalars $\alpha_1, \ldots, \alpha_n$, such that $x = \sum_{i=1}^{n} \alpha_i x_i$
- Fact: all bases of a space have the same number of vectors (the *dimension* of the space).
- Fact: every vector x has a unique representation in a given basis. Thus we can represent a vector by its coefficients, $x = (\alpha_1, \ldots, \alpha_n)$.
- terminology: we say that a basis *spans* a vector space.

F. Definition: given vector spaces V and W, a *linear transformation* is a mapping $A : V \to W$, such that for all $\alpha, \beta \in \Re$ and for all $x, y \in V$,

$$A(\alpha x + \beta y) = \alpha A x + \beta A y.$$

II. Norms

A. Definition: a *norm* on a vector space V is a function $\|\cdot\| : V \to \Re$, such that for all $x, y \in V$ and for all scalars α

$\|x\| \geq 0$

$\|x\| = 0$ iff $x = 0$

$\|\alpha x\| = |\alpha|\ \|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

B. Examples

1. on $\Re^n$

$$\begin{array}{lll} \|x\|_2 & = (x_1^2 + \cdots + x_n^2)^{1/2} & (l_2, \text{Euclidean}) \\ \|x\|_1 & = |x_1| + \cdots + |x_n| & (l_1) \\ \|x\|_\infty & = \max_i |x_i| & (l_\infty, \text{infinity}) \end{array}$$

2. on $\Re^{n \times n}$

- defined in terms of a norm on $\Re^n$.

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

- intuition: maximum amount that a unit-vector is stretched by the linear transformation represented by A.
- examples:

$$\begin{aligned} \|A\|_2 &= \sqrt{\rho(A^T A)}, \quad \text{where } \rho(A^T A) = \text{max eigenvalue of } A^T A \\ \|A\|_1 &= \max_j \sum_{i=1}^n |A_{ij}| \\ \|A\|_\infty &= \max_i \sum_{j=1}^n |A_{ij}| \end{aligned}$$

3. on $V = \{\text{all polynomials}\}$

$$\|p\|_\infty = \max_{0 \leq x \leq 1} |p(x)|$$

III. Inner products

A. Definition: an *inner product* is a function $(\cdot, \cdot) : V \times V \to \Re$, satisfying:

$(x, x) \geq 0$

$(x, x) = 0$ iff $x = 0$

$(x, y) = (y, x)$

$(\alpha x, y) = \alpha(x, y)$

$(x + y, z) = (x, z) + (y, z)$

B. Examples

$V = \Re^n$	$(x, y) = x^T y = \sum_1^n x_i y_i$
$V = P_n = \{\text{polynomials of degree } \leq n\}$	$(f, g) = \sum_1^{n+1} f(x_i) g(x_i)$
$V = \{\text{continuous functions on } [0, 1]\}$	$(f, g) = \int_0^1 f(x) g(x) dx$

C. Orthogonality.

- two vectors x, y are *orthogonal* if $(x, y) = 0$.
- a set $\{x_1, \ldots, x_n\}$ is orthogonal if $(x_i, x_j) = 0, i \neq j$.
- an orthogonal set $\{x_1, \ldots, x_n\}$ is *orthonormal* if $(x_i, x_i) = 1$.

D. Norms derived from inner products.

$\|x\| = \sqrt{(x, x)}$ is a norm.

E. Cauchy-Schwarz inequality:

$$|(x, y)| \leq \|x\|\ \|y\|, \quad \text{where } \|\cdot\| = \sqrt{(\cdot, \cdot)}$$