CS/MATH 3414 Homework # 10

(6) 1. Compute w_i and x_i $(i = 0, 1, 2)$ such that the formula

$$\int_{-1}^{1} f(x) \cos\left(\frac{\pi x}{2}\right)\, dx \approx \sum_{i=0}^{2} w_i f(x_i)$$

is exact if f is a polynomial of degree ≤ 5. Give w_i and x_i to 12 significant digits.

(4) 2. Compute coefficients such that the formula

$$\int_{0}^{\infty} f(x) e^{-x}\, dx \approx w_0 f(0) + \tilde{w}_0 f'(0) + w_2 f(1) + \tilde{w}_2 f'(1)$$

is exact if f is a polynomial of degree ≤ 3. [Hint: $\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n!$]

(2) 3. Compute the first three orthogonal polynomials Ω_0, Ω_1, Ω_2 with respect to the inner product $\langle f, g \rangle = -\int_0^1 f(x) g(x) \ln x\, dx$.

Extra Credit

(3) 4. Express x^5 as a linear combination of the Chebyshev polynomials T_0, T_1, …, T_5:

$$x^5 = \underline{\qquad\qquad} T_0 + \underline{\qquad\qquad} T_1 + \underline{\qquad\qquad} T_2 + \underline{\qquad\qquad} T_3 + \underline{\qquad\qquad} T_4 + \underline{\qquad\qquad} T_5.$$

(5) 5. Find the polynomial $p^*(x)$ of degree ≤ 3 which minimizes

$$\|p(x) - x^5\|^2 = \int_{-1}^{1} \frac{\left(x^5 - p(x)\right)^2}{\sqrt{1 - x^2}} dx$$

over all polynomials $p(x)$ of degree ≤ 3.

LEAST SQUARES APPROXIMATION EXAMPLE

Problem: Find the polynomial $p^*(x)$ of degree ≤ 2 which minimizes

$$\int_{-1}^{1} \big(p(x) - \sin x\big)^2 dx.$$

Solution: Define $\langle r, s\rangle = \int_{-1}^{1} r(x)s(x)\,dx$, $f(x) = \sin x$. Then the problem is equivalent to

$$\min_p \int_{-1}^{1} \big(p(x) - \sin x\big)^2 dx = \min_p \langle p - f, p - f\rangle = \min_p \|p - f\|^2 \tag{1}$$

or

$$\min_p \|p - f\| \tag{2}$$

where the minimization is done over all polynomials p of degree ≤ 2. The solution to (2) is

$$p^* = \sum_{i=0}^{2} \frac{\langle f, \phi_i\rangle}{\langle \phi_i, \phi_i\rangle}\, \phi_i \tag{3}$$

where ϕ_0, ϕ_1, ϕ_2 are orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot\rangle$. Now, working out the details, $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = (3/2)(x^2 - 1/3)$, $\langle \phi_0, \phi_0\rangle = 2$, $\langle \phi_1, \phi_1\rangle = 2/3$, $\langle \phi_2, \phi_2\rangle = 2/5$, $\langle f, \phi_0\rangle = 0$, $\langle f, \phi_1\rangle = 2(\sin 1 - \cos 1)$, $\langle f, \phi_2\rangle = 0$. Thus

$$p^*(x) = 0 \cdot 1 + \frac{2(\sin 1 - \cos 1)}{2/3}\, x + 0 \cdot \frac{3}{2}(x^2 - 1/3) = 3(\sin 1 - \cos 1)x.$$