1. The following algorithm is the most obvious (but far from most efficient) way to evaluate a polynomial \( f(x) = \sum_{i=0}^{N} a_i x^i \). Assume that the coefficients are stored in an array \( a[] \).

\[
\text{Poly} = 0; \quad \text{\text{\quad \quad \quad // 1}}
\]
\[
\text{for (i = 0; i <= N; i++) \{} \quad \text{\quad \quad \quad // 1 before, 2 per pass, 1 to exit}
\]
\[
\text{\quad xtoN = 1;} \quad \text{\quad \quad \quad \quad \quad // 1}
\]
\[
\text{\quad for (j = 1; j <= i; j++) \{} \quad \text{\quad \quad \quad // 1 before, 2 per pass, 1 to exit}
\]
\[
\text{\quad \quad \quad xtoN = x * xtoN;} \quad \text{\quad \quad \quad \quad \quad // 2}
\]
\[
\text{\quad Poly = Poly + xtoN * a[i];} \quad \text{\quad \quad \quad // 4}
\]
\[
\text{\}}
\]

a) [20 points] Derive a complexity function \( T(N) \) for the given algorithm.

\[
T(N) = 1 + 1 + \sum_{i=0}^{N} \left( 2 + 1 + 1 + \sum_{j=1}^{i} (2 + 2) + 1 + 4 \right) + 1
\]
\[
= \sum_{i=0}^{N} \left( \sum_{j=1}^{i} 4 + 9 \right) + 3
\]
\[
= \sum_{i=0}^{N} (4i + 9) + 3
\]
\[
= 4 \frac{N(N+1)}{2} + 9(N+1) + 3
\]
\[
= 2N^2 + 11N + 12
\]

b) [6 points] State the \( \Theta \)-complexity of \( T(N) \).

\( T(N) \text{ is } \Theta(N^2) \)
2. [20 points] Consider the following function, where $\alpha$ is an unknown constant:

$$f(n) = n^\alpha + \log n$$

Use Theorem 8 from the course notes on asymptotics to prove the following fact:

if $\alpha > 0$, $f(n)$ is $\Theta(n^\alpha)$

Show all work to support every step in your proof. Hint: if $\alpha > 0$, then

$$\lim_{n \to \infty} n^\alpha = \infty$$

According to Theorem 8, we need to examine the following limit:

$$\lim_{n \to \infty} \frac{n^\alpha + \log n}{n^\alpha} = \lim_{n \to \infty} \left( \frac{n^\alpha}{n^\alpha} + \frac{\log n}{n^\alpha} \right)$$

algebra

$$= \lim_{n \to \infty} \left( 1 + \frac{\log n}{n^\alpha} \right)$$

algebra

$$= 1 + \lim_{n \to \infty} \left( \frac{\log n}{n^\alpha} \right)$$

limit of a constant

$$= 1 + \lim_{n \to \infty} \left( \frac{1/\ln 2}{\alpha n^{\alpha-1}} \right)$$

l'Hopital's Rule

$$= 1 + \lim_{n \to \infty} \left( \frac{1}{(\alpha \ln 2)n^\alpha} \right)$$

algebra

$$= 1 + 0$$

from given limit factoid

$$= 1$$

So, by Theorem 8, $f(n)$ is $\Theta(n^\alpha)$. 

3. [24 points] Suppose that executing an algorithm on input of size $N$ requires executing $T(N) = 5N \log N + 8N$ instructions.

   a) How long would it take to execute this algorithm on hardware capable of carrying out $2^{20}$ instructions per second if $N = 2^{24}$? (Give your answer in hours, minutes and seconds, to the nearest second.)

   The total number of instructions executed would be:
   
   $$T(2^{24}) = 5 \cdot 2^{24} \log 2^{24} + 8 \cdot 2^{24} = 5 \cdot 2^{24} \cdot 24 \log 2 + 8 \cdot 2^{24} = 128 \cdot 2^{24}$$

   The time required would be:
   
   $$
   \frac{T(2^{24})}{2^{20}} = \frac{128 \cdot 2^{24}}{2^{20}} = 128 \cdot 2^4 = 2048 \text{ seconds}
   $$

   And that is 34 minutes 8 seconds.

   b) Repeat assuming the hardware can execute $2^{22}$ instructions per second.

   Following the same logic, the time required would be:
   
   $$
   \frac{T(2^{24})}{2^{22}} = \frac{128 \cdot 2^{24}}{2^{22}} = 128 \cdot 2^2 = 512 \text{ seconds}
   $$

   And that amounts to 8 minutes and 32 seconds.
4. [30 points] Divide the following functions into categories, so that two functions, say $f(n)$ and $g(n)$, are in the same category if and only if $f(n) \Theta (g(n))$. Arrange the categories from the lowest order of magnitude to the highest. A function may be in a category by itself, or there may be several functions in the same category.

<table>
<thead>
<tr>
<th>Category</th>
<th>Members</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta(1)$</td>
<td>5000</td>
</tr>
<tr>
<td>$\Theta(\log \log n)$</td>
<td>$\log(\log n^2)$</td>
</tr>
<tr>
<td>$\Theta(\log n)$</td>
<td>$\log n$, $\log n^2$</td>
</tr>
<tr>
<td>$\Theta(\log^5 n)$</td>
<td>$\log^5 n$</td>
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<tr>
<td>$\Theta(n^{0.3})$</td>
<td>$n^{0.3}$</td>
</tr>
<tr>
<td>$\Theta(n)$</td>
<td>$n + \log n$, $(n^2 + 4)^{1/2}$, $4n + n^{1/2}$</td>
</tr>
<tr>
<td>$\Theta(n^2)$</td>
<td>$n^2$, $n^2 - 100n$</td>
</tr>
<tr>
<td>$\Theta(n^2 \log n)$</td>
<td>$n^2 \log n$</td>
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<tr>
<td>$\Theta(3^n)$</td>
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