

Let $N \geq 0$, let A , B , and C be constants, and let f and g be any functions. Then:

$$\sum_{k=1}^N C f(k) = C \sum_{k=1}^N f(k)$$

S1: factor out constant

$$\sum_{k=1}^N (f(k) \pm g(k)) = \sum_{k=1}^N f(k) \pm \sum_{k=1}^N g(k)$$

S2: separate summed terms

$$\sum_{k=1}^N C = NC$$

S3: sum of constant

$$\sum_{k=1}^N k = \frac{N(N+1)}{2}$$

S4: sum of k

$$\sum_{k=1}^N k^2 = \frac{N(N+1)(2N+1)}{6}$$

S5: sum of k squared

$$\sum_{k=0}^N 2^k = 2^{N+1} - 1$$

S6: sum of 2^k

$$\sum_{k=1}^N k 2^{k-1} = (N-1)2^N + 1$$

S7: sum of $k 2^{k-1}$

Let b be a real number, $b > 0$ and $b \neq 1$. Then, for any real number $x > 0$, the *logarithm of x to base b* is the power to which b must be raised to yield x . That is:

$$\log_b(x) = y \text{ if and only if } b^y = x$$

For example:

$$\log_2(64) = 6 \text{ because } 2^6 = 64$$

$$\log_2(1/8) = -3 \text{ because } 2^{-3} = 1/8$$

$$\log_2(1) = 0 \text{ because } 2^0 = 1$$

If the base is omitted, the standard convention in mathematics is that log base 10 is intended; in computer science the standard convention is that log base 2 is intended.

Let a and b be real numbers, both positive and neither equal to 1. Let $x > 0$ and $y > 0$ be real numbers.

L1: $\log_b(1) = 0$

L2: $\log_b(b) = 1$

L3: $\log_b(x) < 0$ for all $0 < x < 1$

L4: $\log_b(x) > 0$ for all $x > 1$

L5: $\log_b(b^y) = y$

L6: $b^{\log_b(x)} = x$

L7: $\log_b(xy) = \log_b(x) + \log_b(y)$

L8: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

L9: $\log_b(x^y) = y \log_b(x)$

L10: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Definition:

Let $f(x)$ be a function with domain (a, b) and let $a < c < b$. The *limit of $f(x)$ as x approaches c is L* if, for every positive real number ϵ , there is a positive real number δ such that whenever $|x-c| < \delta$ then $|f(x) - L| < \epsilon$.

The definition being cumbersome, the following theorems on limits are useful. We assume $f(x)$ is a function with domain as described above and that K is a constant.

c1:
$$\lim_{x \rightarrow c} K = K$$

c2:
$$\lim_{x \rightarrow c} x = c$$

c3:
$$\lim_{x \rightarrow c} x^r = c^r \text{ for all } r > 0$$

Here assume $f(x)$ and $g(x)$ are functions with domain as described above and that K is a constant, and that both the following limits exist (and are finite):

$$\lim_{x \rightarrow c} f(x) = A$$

$$\lim_{x \rightarrow c} g(x) = B$$

Then:

$$\mathbf{c4:} \quad \lim_{x \rightarrow c} Kf(x) = K \lim_{x \rightarrow c} f(x)$$

$$\mathbf{c5:} \quad \lim_{x \rightarrow c} (f(x) \pm g(x)) = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x)$$

$$\mathbf{c6:} \quad \lim_{x \rightarrow c} (f(x) * g(x)) = \lim_{x \rightarrow c} f(x) * \lim_{x \rightarrow c} g(x)$$

$$\mathbf{c7:} \quad \lim_{x \rightarrow c} (f(x) / g(x)) = \lim_{x \rightarrow c} f(x) / \lim_{x \rightarrow c} g(x) \text{ provided } B \neq 0$$

Definition:

Let $f(x)$ be a function with domain $[0, \infty)$. The *limit of $f(x)$ as x approaches ∞* is L if, for every positive real number ϵ , there is a positive real number N such that whenever $x > N$ then $|f(x) - L| < \epsilon$.

The definition being cumbersome, the following theorems on limits are useful. We assume $f(x)$ is a function with domain $[0, \infty)$ and that K is a constant.

c8:
$$\lim_{x \rightarrow \infty} K = K$$

c9:
$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

c10:
$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0 \text{ for all } r > 0$$

Given a rational function the last two rules are sufficient if a little algebra is employed:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{7x^2 + 5x + 10}{3x^2 + 2x + 5} &= \lim_{x \rightarrow \infty} \frac{7 + \frac{5}{x} + \frac{10}{x^2}}{3 + \frac{2}{x} + \frac{5}{x^2}} \\ &= \frac{\lim_{x \rightarrow \infty} 7 + \lim_{x \rightarrow \infty} \frac{5}{x} + \lim_{x \rightarrow \infty} \frac{10}{x^2}}{\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \frac{2}{x} + \lim_{x \rightarrow \infty} \frac{5}{x^2}} \\ &= \frac{7 + 0 + 0}{3 + 0 + 0} \\ &= \frac{7}{3} \end{aligned}$$

Divide by highest power of x from the denominator.

Take limits term by term.

Apply theorem C3.

In some cases, the limit may be infinite. Mathematically, this means that the limit does not exist.

$$\text{c11: } \lim_{x \rightarrow \infty} x^r = \infty \text{ for all } r > 0$$

$$\text{c13: } \lim_{x \rightarrow \infty} (e^x) = \infty$$

$$\text{c12: } \lim_{x \rightarrow \infty} (\log_b x) = \infty$$

Example:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{7x^2 + 5x + 10}{2x + 5} &= \lim_{x \rightarrow \infty} \frac{7x + 5 + \frac{10}{x}}{2 + \frac{5}{x}} \\ &= \frac{\lim_{x \rightarrow \infty} 7x + \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{10}{x}}{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{5}{x}} = \infty \end{aligned}$$

In some cases, the reduction trick shown for rational functions does not apply:

$$\lim_{x \rightarrow \infty} \frac{7x + 5 \log(x) + 10}{2x + 5} = ??$$

In such cases, l'Hôpital's Rule is often useful. If $f(x)$ and $g(x)$ are differentiable functions such that

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = \infty$$

then:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

This also applies if the limit is 0.

Applying l'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \frac{7x + 5 \log(x) + 10}{2x + 5} = \lim_{x \rightarrow \infty} \frac{7 + \frac{5}{x}}{2} = \frac{7}{2}$$

Another example:

$$\lim_{x \rightarrow \infty} \frac{x^3 + 10}{e^x} = \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{6x}{e^x} = \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0$$

Recall that: $D[e^{f(x)}] = e^{f(x)} D[f(x)]$

Mathematical induction is a technique for proving that a statement is true for all integers in the range from N_0 to ∞ , where N_0 is typically 0 or 1.

First Principle of Mathematical Induction

Let $P(N)$ be a proposition regarding the integer N , and let S be the set of all integers k for which $P(k)$ is true. If

- 1) N_0 is in S , and
- 2) whenever N is in S then $N+1$ is also in S ,

then S contains all integers in the range $[N_0, \infty)$.

To apply the PMI, we must first establish that a specific integer, N_0 , is in S (establishing the basis) and then we must establish that if a arbitrary integer, $N \geq N_0$, is in S then its successor, $N+1$, is also in S .

Theorem: For all integers $n \geq 1$, n^2+n is a multiple of 2.

proof: Let S be the set of all integers for which n^2+n is a multiple of 2.

If $n = 1$, then $n^2+n = 2$, which is obviously a multiple of 2. This establishes the basis, that 1 is in S .

Now suppose that some integer $k \geq 1$ is an element of S . Then k^2+k is a multiple of 2. We need to show that $k+1$ is an element of S ; in other words, we must show that $(k+1)^2+(k+1)$ is a multiple of 2. Performing simple algebra:

$$(k+1)^2+(k+1) = (k^2 + 2k + 1) + (k + 1) = k^2 + 3k + 2$$

Now we know k^2+k is a multiple of 2, and the expression above can be grouped to show:

$$(k+1)^2+(k+1) = (k^2 + k) + (2k + 2) = (k^2 + k) + 2(k + 1)$$

The last expression is the sum of two multiples of 2, so it's also a multiple of 2. Therefore, $k+1$ is an element of S .

Therefore, by PMI, S contains all integers $[1, \infty)$.

QED

Theorem: Every integer greater than 3 can be written as a sum of 2's and 5's.

(That is, if $N > 3$, then there are nonnegative integers x and y such that $N = 2x + 5y$.)

This is not (easily) provable using the First Principle of Induction. The problem is that the way to write $N+1$ in terms of 2's and 5's has little to do with the way N is written in terms of 2's and 5's. For example, if we know that

$$N = 2x + 5y$$

we can say that

$$N + 1 = 2x + 5y + 1 = 2x + 5(y - 1) + 5 + 1 = 2(x + 3) + 5(y - 1)$$

but we have no reason to believe that $y - 1$ is nonnegative. (Suppose for example that N is 9.)

There is a second statement of induction, sometimes called the "strong" form, that is adequate to prove the result on the preceding slide:

Second Principle of Mathematical Induction

Let $P(N)$ be a proposition regarding the integer N , and let S be the set of all integers k for which $P(k)$ is true. If

- 1) N_0 is in S , and
- 2) whenever N_0 through N are in S then $N+1$ is also in S ,

then S contains all integers in the range $[N_0, \infty)$.

Interestingly, the "strong" form of induction is logically equivalent to the "weak" form stated earlier; so in principle, anything that can be proved using the "strong" form can also be proved using the "weak" form.

Theorem: Every integer greater than 3 can be written as a sum of 2's and 5's.

proof: Let S be the set of all integers $n > 3$ for which $n = 2x + 5y$ for some nonnegative integers x and y .

If $n = 4$, then $n = 2*2 + 5*0$. If $n = 5$, then $n = 2*0 + 5*1$. This establishes the basis, that 4 and 5 are in S .

Now suppose that all integers from 4 through k are elements of S , where $k \geq 5$. We need to show that $k+1$ is an element of S ; in other words, we must show that $k+1 = 2r + 5s$ for some nonnegative integers r and s .

Now $k+1 \geq 6$, so $k-1 \geq 4$. Therefore by our assumption, $k-1 = 2x + 5y$ for some nonnegative integers x and y . Then, simple algebra yields that:

$$k+1 = k-1 + 2 = 2x + 5y + 2 = 2(x+1) + 5y,$$

whence $k+1$ is an element of S .

Therefore, by the Second PMI, S contains all integers $[4, \infty)$.

QED