Turn this assignment in electronically using the Curator system. No late assignments will be accepted.

For Problems 1 and 2, consider the following function: $f(n) = 5n + \frac{3}{2} n \log n + 4$

1. [10 points] Using Theorem 8 from the course notes, prove that f(n) is O(n log n).

Using Theorem 8, we take the limit:

$$\lim_{n\to\infty} \frac{f(n)}{n \log n} = \lim_{n\to\infty} \frac{5n + \frac{3}{2} n \log n + 4}{n \log n} = \lim_{n\to\infty} \frac{5n}{n \log n} + \lim_{n\to\infty} \frac{\frac{3}{2} n \log n}{n \log n} + \lim_{n\to\infty} \frac{4}{n \log n}$$

$$= \lim_{n\to\infty} \frac{5}{\log n} + \lim_{n\to\infty} \frac{3}{2} + \lim_{n\to\infty} \frac{4}{n \log n} = 0 + \frac{3}{2} + 0 = \frac{3}{2}$$

Since 3/2 is a positive constant where $0 < 3/2 < \infty$, we can conclude that f(n) is Θ(n log n). Therefore, f(n) is also O(n log n) by the definition of Big-Θ (i.e. if f(n) is Θ(g(n)), then f(n) is also O(g(n)) by the definition of Big-Θ).

2. [15 points] Write and simplify T(n) expressions and determine the Θ category for the complexity function of each of the following code fragments. Be sure to clearly label and show both your T(n) expression AND your Θ category in your answer.

(a)

```
for (int i = 0; i < n; i++) {
      for (int j = 1; j <= n; j++) {
            x = x + y[j];
}
```

$$T(n) = 1 + \sum_{i=1}^{n} \left[1 + 1 + 1 + \sum_{j=1}^{n} (1 + 1 + 3) + 1 \right] + 1$$

$$= 2 + \sum_{i=1}^{n} \left[4 + \sum_{j=1}^{n} 5 \right] = 2 + \sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \sum_{j=1}^{n} 5$$

$$= 2 + 4n + \sum_{i=1}^{n} 5n = 2 + 4n + 5n^2$$

Θ category = Θ(n^2)

(b)

```
for (int i = 0; i < n; i++) {
      for (int j = 1; j <= i; j++)
            x = x + y[j];
}
```

$$T(n) = 1 + \sum_{i=1}^{n} \left[1 + 1 + 1 + \sum_{j=1}^{i} (1 + 1 + 3) + 1 \right] + 1$$

$$= 2 + \sum_{i=1}^{n} \left[4 + \sum_{j=1}^{i} 5 \right] = 2 + \sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \sum_{j=1}^{i} 5$$

$$= 2 + 4n + 5 * \sum_{i=1}^{n} i = 2 + 4n + \frac{5(n^2 + n)}{2}$$

$$= 2 + \frac{13n}{2} + \frac{5n^2}{2}$$

Θ category = $\Theta(n^2)$

(c)

```
for (int i = 0; i < n; i *= 2) {
      for (int j = 1; j <= n; j++)
            x = x + y[j];
}
```

$$T(n) = 1 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} \left[1 + 1 + 1 + \sum_{j=1}^{n} (1 + 1 + 3) + 1 \right] + 1$$

$$= 2 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} \left[4 + \sum_{j=1}^{n} 5 \right] = 2 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} 4 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} \sum_{j=1}^{n} 5$$

$$= 2 + 4(\lfloor \log n \rfloor + 1) + 5 * \sum_{i=1}^{\lfloor \log n \rfloor + 1} n$$

$$= 2 + 4(\lfloor \log n \rfloor + 1) + 5n(\lfloor \log n \rfloor + 1)$$

$$= 2 + 4 + 4\lfloor \log n \rfloor + 5n + 5n\lfloor \log n \rfloor$$

$$= 6 + 4\lfloor \log n \rfloor + 5n + 5n\lfloor \log n \rfloor$$

Θ category = Θ(n log n)

(d)

```
for (int i = 0; i < n; i *= 2) {
      for (int j = 1; j <= 5; j++)
            x = x + y[j];
}
```

$$T(n) = 1 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} \left[1 + 1 + 1 + \sum_{j=1}^{5} (1 + 1 + 3) + 1 \right] + 1$$

$$= 2 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} \left[4 + \sum_{j=1}^{5} 5 \right] = 2 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} [4 + 25]$$

$$= 2 + \sum_{i=1}^{\lfloor \log n \rfloor + 1} 29$$

$$= 2 + 29(\lfloor \log n \rfloor + 1)$$

$$= 2 + 29 + 29 \lfloor \log n \rfloor$$

$$= 31 + 29 \lfloor \log n \rfloor$$

Θ category = $\Theta(\log n)$

3. [15 points] Divide the following 12 functions into non-overlapping categories (Θ equivalence classes), so that two functions, f(n) and g(n), are in the same category if and only if f(n) is Θ(g(n)). Arrange the categories from the lowest order of magnitude to the highest. A function may be in a category by itself, or there may be several functions in the same category. Clearly label each category and the functions it contains.

$n^{1/2}$	$2n^{0.5}$	2^n
5	$n^{3/4}$	$n^2 \log n$
$2n + \log n$	$5 \log n^2$	$3n^2 - 2n^2 + 1$
$(\log n)^3$	$(n^2 + 2)^{1/2}$	$3n^2 - 3n^2 + 1$

Many of these functions can be categorized by looking at the dominant term and following Theorem 13 from the course notes. For other functions, Theorem 8 from the notes can be used along with a guess.

Here is an example using Theorem 8. Consider the function $(n^2 + 2)^{1/2}$. Guessing that this function may be Θ(n) and applying Theorem 8 gives:

$$\lim_{n\to\infty} \frac{(n^2+2)^{1/2}}{n} = \lim_{n\to\infty}\left(\frac{n^2+2}{n^2}\right)^{1/2} = \lim_{n\to\infty}\left(1+\frac{2}{n^2}\right)^{1/2} = 1$$

which proves that $(n^2 + 2)^{1/2}$ is Θ(n) because 1 is a positive constant $0 < 1 < \infty$

Below is the complete list of categories and the functions contained in each category.

Category	Contains	Comments
$\Theta(1)$	5	
	$3n^2 - 3n^2 + 1 = 1$	(simplifies to a constant)
$\Theta(\log n)$	$5 \log n^2$	
$\Theta((\log n)^3)$	$(\log n)^3$	(use Theorem 8 and l'Hopital's Rule, can show that $(\log n)^3$ is between log n and n)
$\Theta(n^{1/2})$	$n^{1/2}$	
	$2n^{0.5}$	
$\Theta(n^{3/4})$	$n^{3/4}$	
$\Theta(n)$	$2n + \log n$	
	$(n^2 + 2)^{1/2}$	(use Theorem 8, see example above)
$\Theta(n^2)$	$3n^2 - 2n^2 + 1 = n^2 + 1$	(simplifies to $n^2 + 1$)
$\Theta(n^2 \log n)$	$n^2 \log n$	(using Theorem 8 and l'Hopital's Rule, can show that $n^2 \log n$ is between $O(n^2)$ and $O(n^3)$)
$\Theta(2^n)$	2^n	