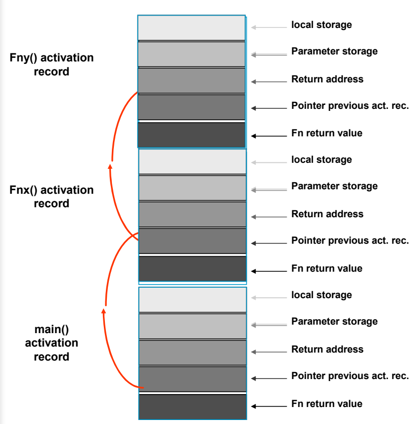
CS 1704

Introduction to Data Structures and Software Engineering

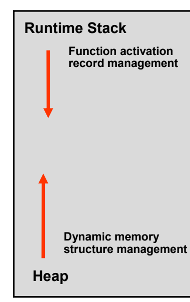
Recursion Underpinnings

- Every instance of a function execution (call) creates an Activation Record, (frame) for the function.
  - Activation records hold required execution information for functions:
    - Return value for the function
    - Pointer to activation record of calling function
    - Return memory address, (calling instruction address)
    - Parameter storage
    - Local variable storage

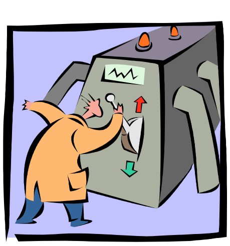


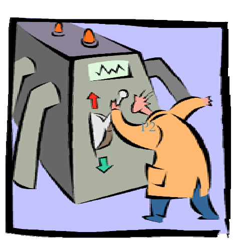
Storage Corruption

- Infinite regression results in a collision between the “run-time” stack & heap termed a “run-time” stack overflow error.
- Illegal pointer de-references (garbage, dangling-references)



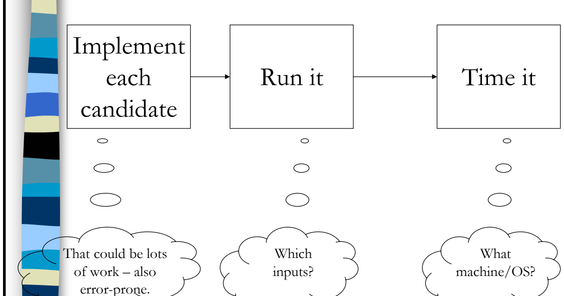
Comparing Algorithms





Should we use Program 1 or Program 2?
Is Program 1 “fast”? “Fast enough”?

The empirical approach



Running Time Implications

- Processor speed differences are too great to be used as a basis for impartial algorithm comparisons.
- Overall system load may cause inconsistent timing results, even if the same compiler and hardware are used.
- Hardware characteristics, such as the amount of physical memory and the speed of virtual memory, can dominate timing results.
- In any case, those factors are irrelevant to the complexity of the algorithm.

Analytical Approach

- Primitive operations
  - x = 4 — assignment
  - ... x + 5 ... — arithmetic
  - if (x < y) ... — comparison
  - x[4] — index an array
  - *x — dereference
  - x.foo( ) — calling a method
- Others
  - new/malloc — memory usage

Rules for Analysis

1. We assume an arbitrary time unit.
2. Running of each of the following type of statement takes time T(1):
   1. assignment statement
   2. I/O statement
   3. Boolean expression evaluation
   4. function return
   5. arithmetic operations
3. Running time of a selection statement (if, switch) is T(1) for the condition evaluation + the maximum of the running times for the individual clauses in the selection.

More Rules

4. Loop execution time is the time for the loop setup (initialization & setup) + the sum, over the number of times the loop is executed, of the body time + time for the loop check and update operations.
   1. Always assume that the loop executes the maximum number of iterations possible
5. Running time of a function call is T(1) for function setup + the time required for the execution of the function body.
6. Running time of a sequence of statements is the largest time of any statement in the sequence.

Summation Formulae

S1: factor out constant

$$\boxed{\sum_{k=1}^{N} Cf(k) = C\sum_{k=1}^{N} f(k)}$$

S3: sum of constant

$$\boxed{\sum_{k=1}^{N} C = NC}$$

S2: separate summed terms

$$\boxed{\sum_{k=1}^{N} (f(k) \pm g(k)) = \sum_{k=1}^{N} f(k) \pm \sum_{k=1}^{N} g(k)}$$

S4: sum of *k*

$$\boxed{\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}}$$

S5: sum of *k* squared

$$\boxed{\sum_{k=1}^{N} k = \frac{N(N+1)}{2}}$$

How many foos?

```
for (j = 1; j <= N; ++j) {
    foo( );
}
```

$$\sum_{j=1}^{N} 1 = N$$

How many foos?

```
for (j = 1; j <= N; ++j) {
        for (k = 1; k <= M; ++k) {
                foo( );
        }
}
```

$$\sum_{j=1}^{N} \sum_{k=1}^{M} 1 = NM$$

How many foos?

```
for (j = 1; j <= N; ++j) {
        for (k = 1; k <= j; ++k) {
                foo( );
        }
}
```

$$\sum_{j=1}^{N} \sum_{k=1}^{j} 1 = \sum_{j=1}^{N} j = \frac{N\,(N+1)}{2}$$

How many foos?

```
void foo(int N) {
        if(N <= 2)
                return;
        foo(N / 2);
}
```

$$T(0) = T(1) = T(2) = 1$$
$$T(n) = 1 + T(n/2) \text{ if } n > 2$$

$$\begin{aligned} T(n) &= 1 + (1 + T(n/4)) \\ &= 2 + T(n/4) \\ &= 2 + (1 + T(n/8)) \\ &= 3 + T(n/8) \\ &= 3 + (1 + T(n/16)) \\ &= 4 + T(n/16) \\ &\ldots \\ &\approx \log_2 n \end{aligned}$$

How many foos?

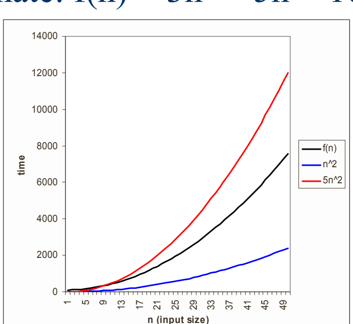
```
for (j = 0; j < N; ++j) {
        for (k = 0; k < j; ++k) {
                foo( );
        }
}
```

$N(N + 1)/2$

```
int N=M;
for (j = 0; j < N; ++j) {
        for (k = 0; k < M; ++k) {
                foo( );
        }
}
```

N^2

Estimate: $f(n) = 3n^2 + 5n + 100$



Function Estimation

If $n \geq 10$ then $n^2 \geq 100$
If $n \geq 5$ then $n^2 \geq 5n$
Therefore, if $n \geq 10$ then:
$f(n) = 3n^2 + 5n + 100 < 3n^2 + n^2 + n^2 = 5n^2$

- So 5n^2 forms an "upper bound" on f(n) if n is 10 or larger (asymptotic bound). In other words, f(n) doesn't grow any faster than 5n^2 "in the long run".

Big-Oh Defined

- To say $f(n)$ is $O(g(n))$ is to say that $f(n)$ is "less than or equal to" $Cg(n)$
- More formally, Let f and g be functions from the set of integers (or the set of real numbers) to the set of real numbers. Then f(x) is said to be O(g(x)), which is read as f(x) is big-oh of g(x), if and only if there are constants C and n0 such that

  | f(x) | <= C | g(x) | whenever x > n0.
- Don't be confused …
  - "$f(n)$ is of **Order** $g(n)$"

The trick

$N(N + 1)/2 \quad + N^2$

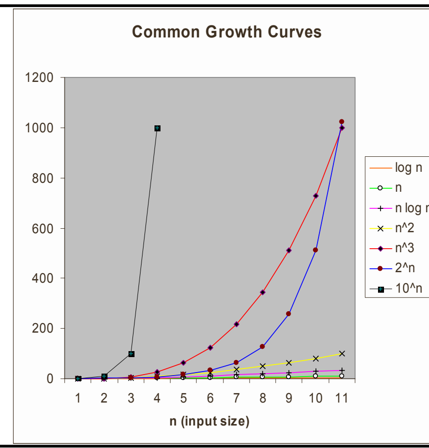
$an^k + bn^{k-1} + \ldots + yn + z$

an^k

n^k

Some complexity classes …

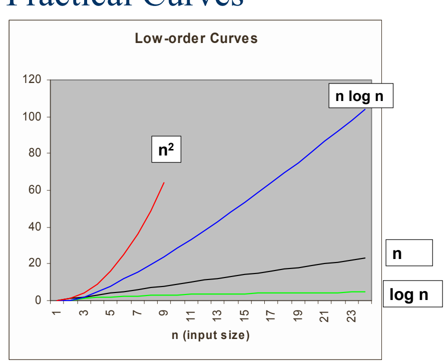
Constant	$O(1)$
Logarithmic	$O(\log n)$
Linear	$O(n)$
Quadratic	$O(n^2)$
Cubic	$O(n^3)$
Polynomial	$O(n^p)$
Exponential	$O(a^n)$



Does it matter?
Let *n* = 1,000, & 1 ms / operation.

	n = 1000, 1 ms/op	max n in one day (first day)
n	1 second	86,400,000
$n \log_2 n$	10 seconds	3,943,234
n^2	17 minutes	9,295
n^3	12 days	442
n^4	32 years	96
n^{10}	3.17×10^{19} years	6
2^n	1.07×10^{301} years	26

Practical Curves



Example

- Assume:
  - 1 day = 100,000 sec (10^5)
    - (actually 86, 400)
  - Input size n = 10^6
  - A computer that executes 1,000,000 (10^6) Instructions/sec
    - C/C++ statement instructions

Comparison

Order: n^2
$(10^6)^2$ Instructions
10^{12} Instructions
$10^{12} / 10^6$ secs to run
10^6 secs to run
$10^6 / 10^5$ days to run
10 days to run

Order: $n \log_2 n$
$10^6 \log_2 10^6$ Instructions
$20 (10^6) = 2 (10^7)$
$2 (10^7) / 10^6$ secs to run
20 sec to run

Question?

- Does the fact that hardware is always becoming faster hardware mean that algorithm complexity doesn't really matter?
- Suppose we could obtain a machine that was capable of executing 10 times as many instructions per second (so roughly 10 times faster than the machine hypothesized on the previous slides).
- How long would the order n^2 algorithm take on this machine with an input size of 10^6?

Doing the Numbers

Order: n^2	
# instructions:	$(10^6)^2 = 10^{12}$
# seconds to run:	$10^{12} / 10^7 = 10^5$
# days to run:	$10^5 / 10^5 = 1$

- Impressed? You shouldn't be. That's still 1 (instead of 20 on the slower machine) day versus 20 seconds if an algorithm of order n log(n) were used.
- What about 100 times faster hardware? 2.4 hours.

Big-Oh Defined

- To say $f(n)$ is $O(g(n))$ is to say that $f(n)$ is "less than or equal to" $Cg(n)$
  - Where C is some constant
  - f(n) = 3n^2 + n + 5
  - C=5
  - g(n)=n^2
- Don't be confused …
  - "$f(n)$ is of **Order** $g(n)$"

Comparison

Order: n^2
$(10^6)^2$ Instructions
10^{12} Instructions
$10^{12} / 10^6$ secs to run
10^6 secs to run
$10^6 / 10^5$ days to run
10 days to run

Order: $n \log_2 n$
$10^6 \log_2 10^6$ Instructions
$20 (10^6) = 2 (10^7)$
$2 (10^7) / 10^6$ secs to run
20 sec to run

What about $n^2 + c$? What about $n^2 + n \log_2 n$?

What about $n^2 + n$? What about cn^2?

Observations

- Within complexity classes the differences between algorithms due to constants of proportionality, (coefficients & lesser terms), are not significant enough to warrant reporting
- Exception: certain (high usage) helper algorithms (e.g., sorting, searching)
  - Because they are used many times
  - Think about a trip to NOVA with cars that drive 60 and 70+ mph respectively, one trip vs. weekly trips

Observations

- Even for moderately small input sizes, Order n^2 algorithms will require FAR more time than Order n log(n) algorithms.
- Large problems with **O**rder > n log(n) cannot practically be executed
  - For n = 1000 (medium problems) n^2 algorithms can still be used

General Rules

- A normal loop has big Oh, O(n)
- A doubly nested loop has big Oh, $O(n^2)$
- A triply nested loop has big Oh, $O(n^3)$
- You can get better times, e.g. O(log n)
  - Binary Search is O(log n)
  - Anytime anything is halved on each iteration, you usually get O(log n)
- Why isn't Merge Sort O(log n)?

The trick

$N(N + 1)/2 \quad + N^2$

$$an^k + bn^{k-1} + \ldots + yn + z$$

$$an^k$$

$$n^k$$

Best Case Analysis

- Assumes the input, data etc. are arranged in the most advantageous order for the algorithm, i.e. causes the execution of the fewest number of instructions.
  - E.g., sorting - list is already sorted; searching - desired item is located at first accessed position.

Worst Case Analysis

- Assumes the input, data etc. are arranged in the most disadvantageous order for the algorithm, i.e. causes the execution of the largest number of statements.
  - E.g., sorting - list is in opposite order; searching - desired item is located at the last accessed position or is missing.

Average Case Analysis

- Determines the average of the running times over all possible permutations of the input data.
  – E.g., searching - desired item is located at every position, for each search), or is missing.

Big-Omega

- Definition: Let f and g be functions from the set of integers (or the set of real numbers) to the set of real numbers. Then f(x) is said to be Ω(g(x)) , which is read as f(x) is big-omega of g(x), if there are constants C and n0 such that | f(x) | >= C | g(x) | whenever x > n0.
- Finds order of "best case"

Big-Theta

- Definition: Let f and g be functions from the set of integers (or the set of real numbers) to the set of real numbers. Then f(x) is said to be Θ(g(x)), which is read as f(x) is big-theta of g(x), if f(x) is O(g(x)), and Ω(g(x)).
- In other words, if Big Oh = Big Omega

Example Of Big Theta

- Consider the function f(x) = 5x^3 + x^2 + 1/(1+x^2). Without going through the complete details on the proof, it's apparent that f is O(x^3), since f(x) <= 7x^3 for x >= 1
- f is Ω(x^3), since f(x) >= 5x^3 for x >= 1
- Hence f is both O(x^3) and Ω(x^3), and thereby f is Θ(x^3) also.

Big Oh? (Foo takes C operations)

```
for (j = 1; j <= N; ++j) {
      foo( );
}
```

$$\sum_{j=1}^{N} 1 = N$$

Big Oh?

```
//We know N>M
for (j = 1; j <= N; ++j) {
      for (k = 1; k <= M; ++k) {
            foo( );
      }
}
```

$$\sum_{j=1}^{N} \sum_{k=1}^{M} 1 = O(NM) = O(N^2)$$

Big Oh?

```
for (j = 1; j <= N; ++j) {
        for (k = 1; k <= j; ++k) {
                foo( );
        }
}
```

$$\sum_{j=1}^{N} \sum_{k=1}^{j} 1 = \sum_{j=1}^{N} j = \frac{N(N+1)}{2} = O(n\text{^}2)$$

Big Oh?

```
int foo(int N) {
 if(N <= 2) return 0;
 return foo(N / 2);
}
```

$T(0) = T(1) = T(2) = 1$

$T(n) \quad = 1 + T(n/2) \text{ if } n > 2$

$$\begin{aligned} T(n) &= 1 + (1 + T(n/4)) \\ &= 2 + T(n/4) \\ &= 2 + (1 + T(n/8)) \\ &= 3 + T(n/8) \\ &= 3 + (1 + T(n/16)) \\ &= 4 + T(n/16) \\ &\ldots \\ &\approx O(\log_2 n) \end{aligned}$$

Big Oh?

```
for (j = 0; j < N; ++j) {
        for (k = 0; k < j; ++k) {
                foo( );
        }
}
int N=M;
for (j = 0; j < N; ++j) {
        for (k = 0; k < M; ++k) {
                foo( );
        }
}
```