Sorting

Slides

Introduction

Definitions and Terminology:

- Internal sorts holds all data in RAM
- External sorts use Files
- Ascending Order:
  - † Low to High
- Descending Order:
  - † High to Low
- Stable Sort:
  - † Maintains the relative order of equal elements, in situ.
  - † Desirable if list is almost sorted or if items with equal values are to also be ordered on a secondary field.

Program efficiency

- Overall program efficiency may depend entirely upon sorting algorithm => clarity must be sacrificed for speed.

Sorting Algorithm Analysis

- Performed upon the “overriding” operation in the algorithm:
  - † Comparisons
  - † Swaps

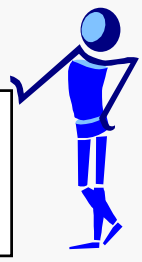
```
void swap( int& x, int& y) {
   int tmp = x;
   x = y;
   y = tmp;
}
```

Bubble Sort

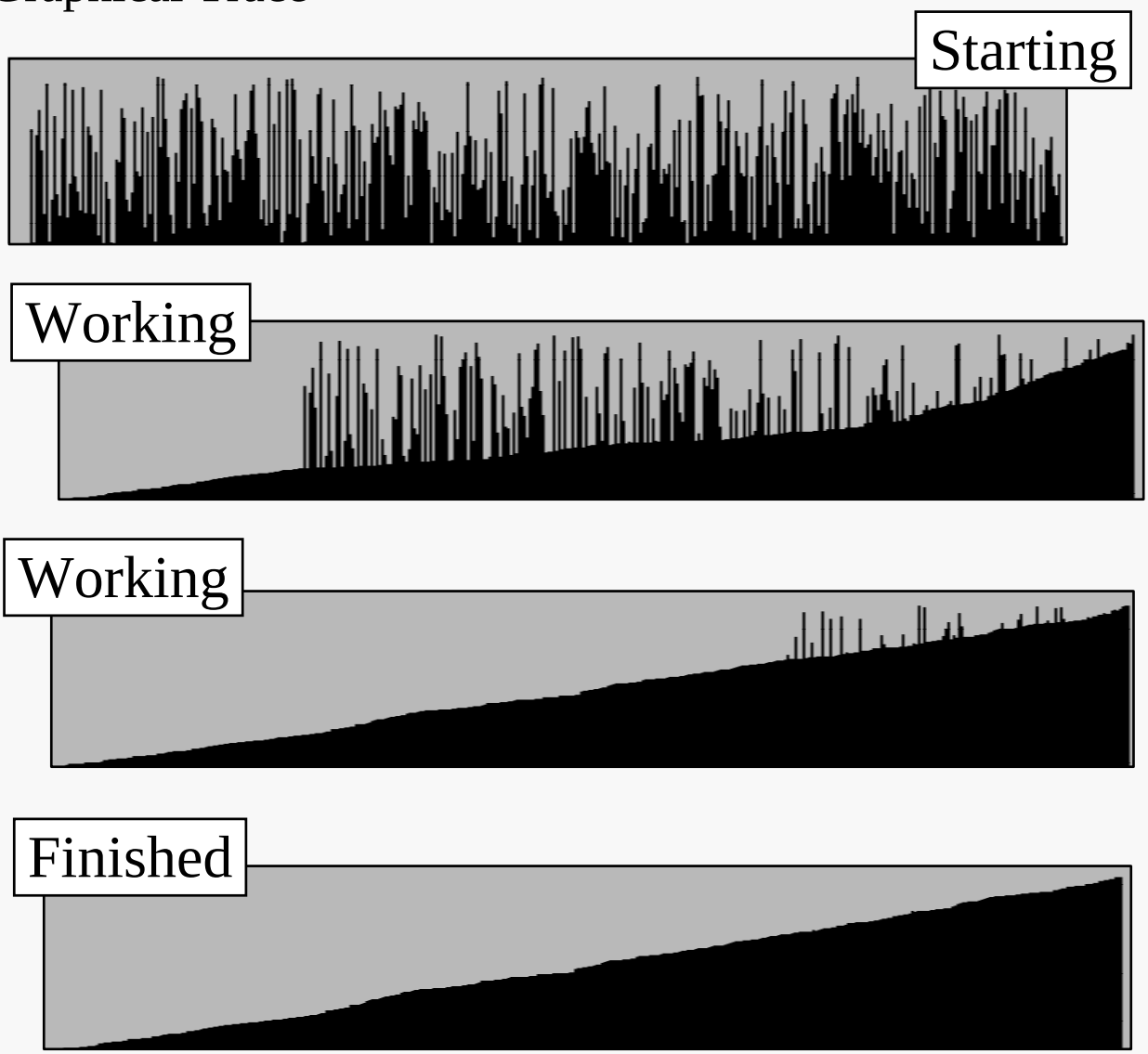
Behavior

- *Bubble* elements down (up) to their location in the sorted order.

```
for (i = 0; i < n-1; i++)
     for (j = n-1; j > i; j--)
          if (A[j] < A[j-1])
               swap(A[j],A[j-1]);
```



Graphical Trace



Bubble Sort Complexity

```
void BubbleSort(int A[], const int n) {

   for (int i = 0; i < n-1; i++)

      for (int j = n-1; j > i; j--) {

         if (A[j] < A[j-1])
            swap(A[j],A[j-1]);
      }
}
```

Time Analysis

if-statement:

time 2 for the condition (and subtraction) + time 4 for the swap (3 assigns + setup), so 5

inner for-loop:

body executed for j-values from n-1 down to i+1, or n-i-1 times

loop body is time 6 for if-stmt + time 2 for test and update

$$\sum_{j=1}^{n-i-1} 8 = 8(n-i-1)$$

outer for-loop:

body executed for i-values from 0 up to n-2 (or 1 to n-1)

loop body is time for inner for-loop + time 3 for loop test and update

So counting the initialization & pre-tests of the for-loop, the overall complexity is:

$$3+\sum_{i=1}^{n-1}\left(3+2+8(n-i-1)\right)=3+\sum_{i=1}^{n-1}\left(8n-8i-3\right)$$

$$=3+8n(n-1)-\frac{8}{2}(n-1)(n-2)-3(n-1)$$

which is clearly $O(n^2)$.

Bubble Sort Complexity

```
void BubbleSort(int A[], const int n) {

   for (int i = 0; i < n-1; i++)

      for (int j = n-1; j > i; j--) {

         if (A[j] < A[j-1])
            swap(A[j],A[j-1]);
      }
}
```

Swaps and Compares Analysis

if-statement:

1 compare and in worst case, 1 swap

inner for-loop:

body executed for j-values from n-1 down to i+1, or n-i-1 times

each execution of body involves 1 compare and up to 1 swap

outer for-loop:

body executed for i-values from 0 up to n-2 (or 1 to n-1)

each execution of body involves n-i-1 compares and up to n-i-1 swaps

So in the worst case, the number of swaps equals the number of compares, and is:

$$\sum_{i=1}^{n-1}(n-i-1) = n(n-1) - \frac{1}{2}(n-1)(n-2) - (n-1)$$

which is clearly $O(n^2)$.

Selection Sort

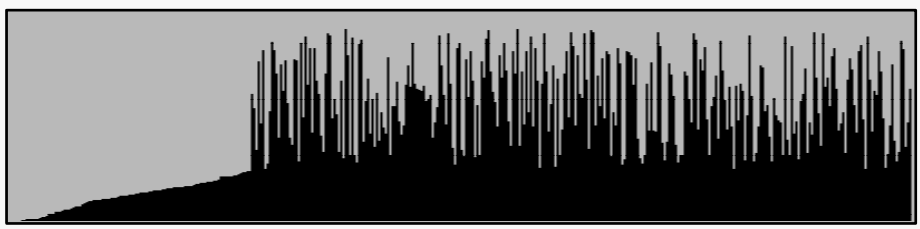
Behavior

- In the i^{th} pass, select the element with the lowest value among A[i], ..., A[n-1], & swap it with A[i].
- Results after i passes: the i lowest elements will occupy A[0], ..., A[i] in sorted order.

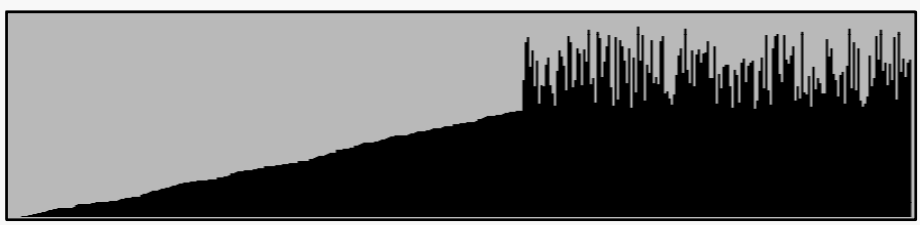
```
for (Begin = 0; Begin < Size - 1; Begin++) {
   SmallSoFar = Begin;
   for (Check = Begin + 1; Check < Size; Check++) {
      if (aList[Check] < aList[SmallSoFar])
         SmallSoFar = Check;
   }
   swap(aList[Begin], aList[SmallSoFar]);
}
```

Graphical Trace

Working



Working



Finished

Selection Sort Complexity

```
void SelectionSort(int aList[] , const int Size)  {
   int Begin, SmallSoFar, Check;

   for (Begin = 0; Begin < Size - 1; Begin++) {
      SmallSoFar = Begin;
      for (Check = Begin + 1; Check < Size; Check++) {
         if (aList[Check] < aList[SmallSoFar])
            SmallSoFar = Check;
      }
      swap(aList[Begin], aList[SmallSoFar]);
   }
}
```

Swaps and Compares Analysis

if-statement: 1 compare

inner for-loop:

body executed n-i-1 times (i is Begin and n is Size)

each execution of body involves 1 compare and no swaps

outer for-loop:

body executed n-1 times

each execution of body involves n-i-1 compares and 1 swap

So in the worst case, the number of swaps is n – 1, and the number of compares is:

$$\sum_{i=1}^{n-1}(n-i-1) = n(n-1) - \frac{1}{2}(n-1)(n-2) - (n-1)$$

which is clearly $O(n^2)$ and the same as for BubbleSort.

Duplex Selection Sort

Min / Max Sorting

- algorithm passes thru the array locating the min and max elements in the array A[i], ..., A[n-i+1]. Swapping the min with A[i] and the max with A[n-i+1].
- Results after the i^{th} pass: the elements A[1], ..., A[i] and A[n-i+1], ..., A[n] are in sorted order.

Unsorted Array

1	2	3	4	5	6	7	8	9	10
95	77	63	58	42	37	31	26	19	12

After 1st Pass

1	2	3	4	5	6	7	8	9	10
12	77	63	58	42	37	31	26	19	95

After 3rd Pass

1	2	3	4	5	6	7	8	9	10
12	19	26	58	42	37	31	63	77	95

5 passes to required sort the above array = N / 2.

Duplex Selection Sort (cont)

Code

```
void DuplexSelectSort(raytype ray, int start,
                      int finish)  {
  int low, high, min, max, small, large, minmax;
  Item tmp;

  low = start;       high = finish;

  while ( low < high ) {
     small = ray[low ]; min = low;
     large = ray[high]; max = high;

     //search for smallest & largest
     for (minmax = low; minmax <= high; minmax++) {
        if (ray [ minmax ] < small) {
           min = minmax;  small = ray [ minmax ];
                 }
        else if (ray [ minmax ] > large) {
           max = minmax;  large = ray [ minmax ];
        }
     } // for minmax

     //check for swap interference
     if ( (max == low) && (min == high) ) {
        swap( ray[low], ray[high] );
     } //check for low 1/2 interference
     else if (max == low) {
        swap( ray[max], ray[high] );
        swap( ray[low], ray[min] );
     } // (min == low) || //no interference
     else {
        swap( ray[min], ray[low] );
        swap( ray[max], ray[high] );
     }
     low++;
     high--;
  } // while
}
```

Recursive implementation: slide 9.14

Comparison Order Analysis

Comparison **O**(DuplexSelectSort)

- Outer Loop : WHILE i loop

  loop limits = 0 . . . N/2-1

  shifted limits = 1 . . . N/2

- Assume subset of array to sort is from 1 .. N

  = 1 . . . N/2 (i.e. start ... finish)

- Inner Loop : FOR j loop

Pass (i)	loop limits	shifted limits
1st	1 . . N	= 1 . . N
2nd	2 . . N - 1	= 1 . . N - 2
3rd	3 . . N - 2	= 1 . . N - 4
• • •	• • •	• • •
ith iteration	i . . N- (i-1)	= 1 . . N-(i-1)*2
• • •	• • •	• • •
N/2-1	= N/2-1 . . N - (N/2-1-1)	
	= N/2-1 . . N/2+2	= 1 . . 4
	= 1 . . N-(i-1)*2	
N/2	= N/2 . . N - (N/2-1)	
	= N/2 . . N/2 + 1	= 1 . . 2

Comparison Order Analysis (cont)

Worst Case:

2 Comparisons on each element

$$= \sum_{i=1}^{N/2} \left[\sum_{j=i}^{N-(i-1)} 2 \right] = \sum_{i=1}^{N/2} \left[\sum_{j=1}^{N-2(i-1)} 2 \right]$$

$$= 2 \sum_{i=1}^{N/2} \left[N - 2(i-1) \right]$$

expanding yields:

= 2[N+N-2+N-4+ … +6+4+2]

summing the arithmetic sequence yields:

$$= 2\left[\frac{1}{2} \left(\frac{N}{2} \left[N + 2 \right] \right) \right]$$

$$= \frac{N^2}{2} + N \in \mathbf{O}(N^2)$$

distributing yields:

$$= 2\left(\frac{N}{2} N \right) - 4\left(\frac{\frac{N}{2}\left[\frac{N}{2} + 1 \right]}{2} - \frac{N}{2} \right)$$

$$= N^2 - 4\left(\frac{\frac{N^2}{4} + \frac{N}{2}}{2} - \frac{N}{2} \right)$$

$$= N^2 - 4\left(\frac{N^2}{8} + \frac{N}{4} - \frac{N}{2} \right)$$

$$= N^2 - \frac{N^2}{2} - N + 2N$$

$$= \frac{N^2}{2} + N \in \mathbf{O}(N^2)$$

Time Analysis

Assume: Start = 1 Finish = N

Analysis: (worst case)

Initialization of High and Low = 2

While Loop

pretest condition = 1

loop body executes N/2 times (i from 1 to N/2)

time cost is 7 (for while condition + assignments + increments) + time for for-loop + time for swap code

For-loop

initialization & pretest condition = 2

loop body executes N-2 (i-1) times

time cost is 2 (for test and increment) + time for if

time cost of if is 4 (two if-conditions + 2 assignments)

Swap Code

time cost is 3 for first if-condition + 1 for second if-condition + cost of two swaps

1 for function setup + time cost of a swap is 3 for assignments

$$T(N) = 3 + \sum_{i=1}^{N/2} \left((7+2) + \sum_{j=1}^{N-2(i-1)} 6 + (3+1+4+4) \right)$$

$$= 3 + \sum_{i=1}^{N/2} \left(21 + 6\left(N - 2(i-1)\right)\right)$$

$$= 3 + \sum_{i=1}^{N/2} \left(6N + 33 - 12i\right)$$

$$= \frac{3}{2} N^2 + \frac{27}{2} N + 3$$

So Duplex Selection Sort is another $O(N^2)$ sort algorithm.

But, the coefficient IS better than for BubbleSort.

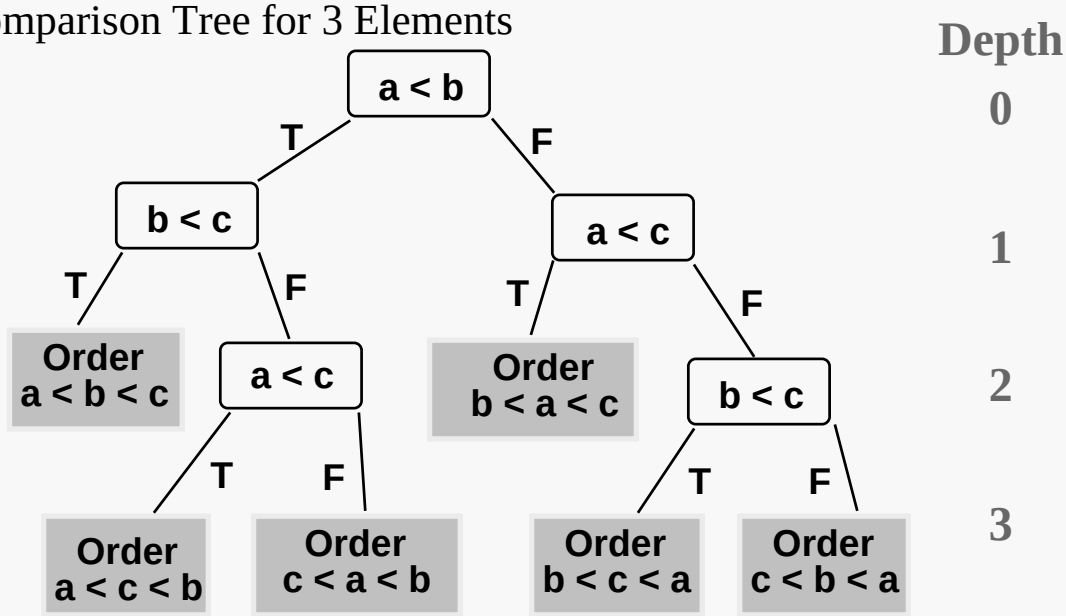
Minimal Comparisons

Comparison-Based Sorting

- Algorithms which compare element values to each other
- What is the minimum number of comparisons, required to sort N elements using a comparison-based sort?

Comparison Tree

- **Binary** tree (hierarchical graph ≤ 2 branches per node) which contains comparisons between 2 elements at each non-leaf node & containing element orderings at its leaf (terminal) nodes.
- Comparison Tree for 3 Elements



3 – Any of the 3 elements (a, b, c) could be first in the final order. Thus there are 3 distinct ways the final sorted order could start.

2 – After choosing the first element, there are two possible selections for the next sorted element.

1 – After choosing the first two elements there is only 1 remaining selection for the last element.

- Therefore selecting the first element one of 3 ways, the second element one of 2 ways and the last element 1 way, there are 6 possible final sorted orderings = **3 * 2 * 1 = 3!**

Comparison Tree Argument

Order Tree for sorting N Elements

- Any of the N elements (1 ... N) could be first in the final order. Thus there are N distinct ways the final sorted order could start.
- After choosing the first element, there are N-1 possible selections for the next sorted element.
- After choosing the first two elements there N-2 possible selections for the next sorted element, etc.
- Therefore selecting the first element one of N ways, the second element one of N-1 ways, etc., there are N * (N-1) * (N-2) * . . . * 2 * 1 possible final sorted orderings which = **N!**

General Comparison Tree Sorting

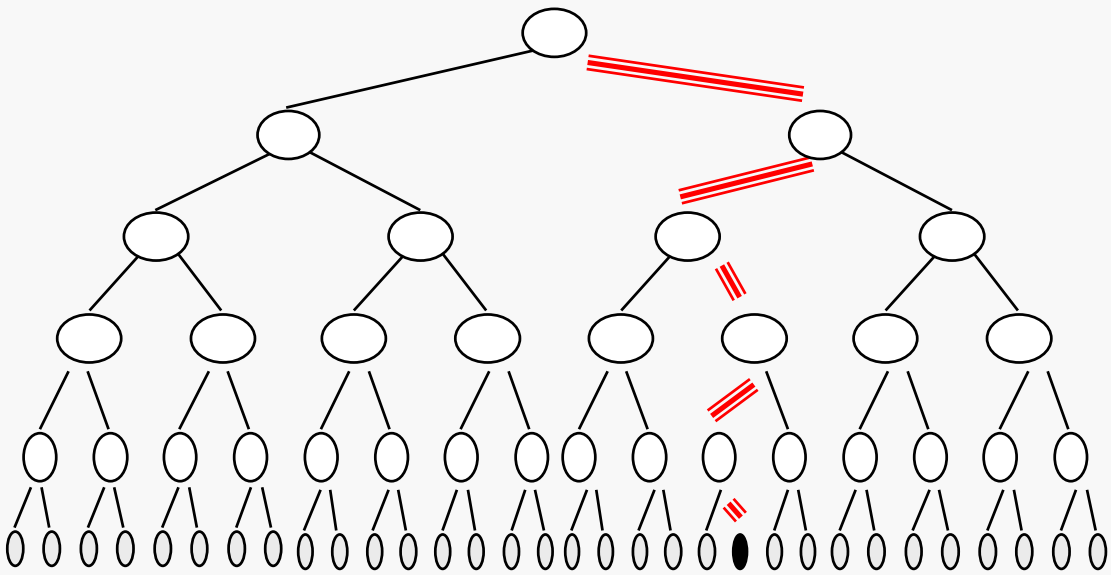
- The comparison tree for N elements must have **N!** leaf nodes. Each leaf node contains one of the possible orderings of all of the N elements.
- Consider the previous comparison tree for 3 elements, all of the leaf nodes are at a **depth** of either 2 or **3** > $\lfloor \log_2 \mathbf{3!} \rfloor$
- The comparison tree for 4 elements must contain **4!**=24 leaf nodes, all of which would be at a depth of either 4 or **5** > $\lfloor \log_2 \mathbf{4!} \rfloor$

floor y= $\lfloor x \rfloor$, y is the largest integer such that y≤ x.

Minimal Order: Comparison Sorting

General Comparison Trees

- The comparison tree for N elements must contain **N!** leaf nodes, all of which would be at a depth > $\lfloor \log_2 \mathbf{N!} \rfloor$
- The minimal number of comparisons required to sort a specific (unsorted) ordering is equal to the depth from the root to a leaf.



- Since the depth of all leaf nodes is > $\lfloor \log_2 \mathbf{N!} \rfloor$ in a comparison tree, the minimal number of comparisons to sort a specific initial ordering of N elements is > $\lfloor \log_2 \mathbf{N!} \rfloor$
- Stirling's Approximation for $\log_2(\mathbf{N!})$ can be used to determine a lower bound for $\log_2(\mathbf{N!})$ which is $\mathbf{O(NlogN)}$
- No comparison based sorting algorithm can sort faster than

∴ $\mathbf{O(N \log N)}$

Quicksort: partitioning

Algorithm

- Select an item in the array as the pivot key.
- Divide the array into two partitions: a left partition containing elements < the pivot key and a right partition containing elements ≥ the pivot key.

Trace

Start with i and j pointing to the first & last elements, respectively.

Select the pivot (3): [3 1 4 1 5 9 2 6 5 8]
R L

Swap the end elements, then move l, r inwards.

[8 1 4 1 5 9 2 6 5 3]
L R

Swap, and repeat: [2 1 4 1 5 9 8 6 5 3]
L R

Swap, and repeat: [2 1 1 | 4 5 9 3 6 5 3]
R L

Partition Function:

```
int Partition(Item A[], int start, int end, Item pivot ){
   int L = start, R = end;

   do {
      swap( A[L] , A[R] );
      while (A[L] < pivot )    L++;
      while (!(A[R] < pivot))  R--;
   } while (R > L);
   return (L);
}
```

Quicksort: find pivot

Pivoting

- Partitioning test requires at least 1 key with a value < that of the pivot, and 1 value ≥ to that of the pivot.
- Therefore, pick the greater of the first two distinct values (if any).

```
const int  MISSING = -1;

int FindPivot(const Item A[], int start, int end ) {

   Item firstkey;  //value of first key found
   int pivot;      //pivot index
   int k;          //run right looking for other key

   firstkey = A[start];
   //return -1 if different keys are not found
   pivot = MISSING;
   k = start + 1;
   //scan for different key
   while ( (k <= end) && (pivot == MISSING) )
      if (firstkey < A[k])      //select key
         pivot = k;
      else if (A[k] < firstkey)
         pivot = start;
      else
         k++;
   return pivot;
}
```

Improving FindPivot

- Try and pick a pivot such that the list is split into equal size sublists, (a speedup that should cut the number of partition steps to about 2/3 that of picking the first element for the pivot).
  - † Choose the middle (median) of the first 3 elements.
  - † Pick k elements at random from the list, sort them & use the median.
- There is a trade-off between reduced number of partitions & time to pick the pivot as k grows.

Recursive Quicksort

Quicksort Function (recursive)

```
const int  MISSING = -1;
void QuickSort( Item A[], int start, int end ) {
   // sort the array from start ... end
   Item pivotKey;
   int  pivotIndex;
   int  k;        //index of partition >= pivot

   pivotIndex = FindPivot( A, start, end );
   if (pivotIndex != MISSING) {
      pivotKey = A[pivotIndex];
      k = Partition( A, start, end, pivotKey );
      QuickSort( A, start, k-1 );
      QuickSort( A, k, end );
   }
}
```

Ave

- quicksort is based upon the intuition that swaps, (moves), should be performed over large distances to be most effective
- quicksort's average running time is faster than any currently known $O(n\log_2 n)$ internal sorting algorithms (by a constant factor).
- For very small n (e.g., $n \leq 16$) a simple $O(n^2)$ algorithm is actually faster than Quicksort.
- When the sublist is small, use another sorting algorithm.

Worst Case = $\mathbf{O(N^2)}$

- In the worst case, every partition might split the list of size j - i + 1 into a list with 1 element, and a list with j - i elements.
- A partition is split into sublists of size 1 & j-i when one of the first two items in the sublist is the largest item in the sublist which is chosen by findpivot.
- When will this worst case partitioning always occur?

Iterative Quicksort

Iterative Conversion

- Iterative implementation requires using a stack to store the partition bounds remaining to be sorted.
- Assume a stack implementation of elements consisting of two integers:

```
struct StackItem {
   int low, hi;
};
```

Partitioning

- At the end of any given partition, only one subpartition need be stacked.
- The second subpartition (equated to the second recursive call), need not be stacked since it is immediately used for the next subpartitioning.

Stacking

- The order of the recursive calls, (i.e., the sorting of the subpartitions) may be made in any order.
- Stacking the larger subpartition assures that the size of the stack is minimized, since the smaller subpartition will be further divided less times than the larger subpartition.

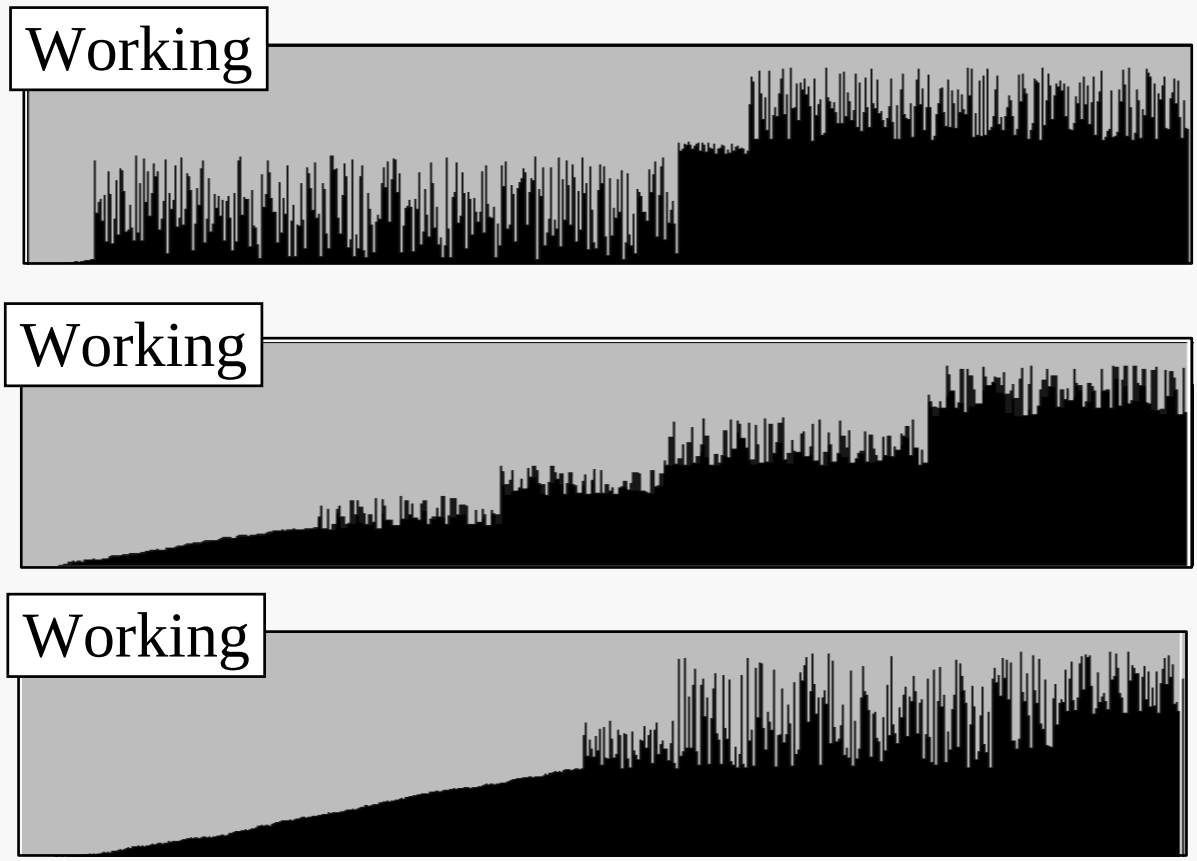
Iterative Quicksort Code

Quicksort Function (iterative)

```
void QuickSort( Item A[], int start, int end ) {
  // sort the array from start ... end
   Item pivotKey;
   int pivotIndex, tmpBnd;
   int k;                    //index of partition >= pivot
   StackItem parts;
   Stack subParts;

   parts.low = start;   parts.hi = end;
   subParts.Push ( parts );

   while ( ! subParts.Empty() )  {
      parts = subParts.Pop();

      while ( parts.hi > parts.low ) {
         pivotIndex = FindPivot( A, parts.low, parts.hi );

         if (pivotIndex != MISSING) {
            pivotKey = A[pivotIndex];
            k = Partition( A, parts.low , parts.hi , pivotKey );
            // push the larger subpartition

            if ( (k-parts.low)  > (parts.hi-k) ) { //stk low part
               tmpBnd = parts.hi;
               parts.hi = k-1;
               subParts.Push( parts );
               parts.low = k;    //set current part to upper part
               parts.hi = tmpBnd;
            } //end  if
            else { // stack upper (larger) part
               tmpBnd = parts.low;
               parts.low = k;
               subParts.Push( parts );
               parts.low = tmpBnd; //set current part to low part
               parts.hi = k-1;
            } // end else
         } // end if

         else          // halt inner loop when all elements equal
            parts.hi = parts.low;
      } // end while

   } // end while

} // end QuickSort
```

Quicksort Efficiency

Graphical Trace



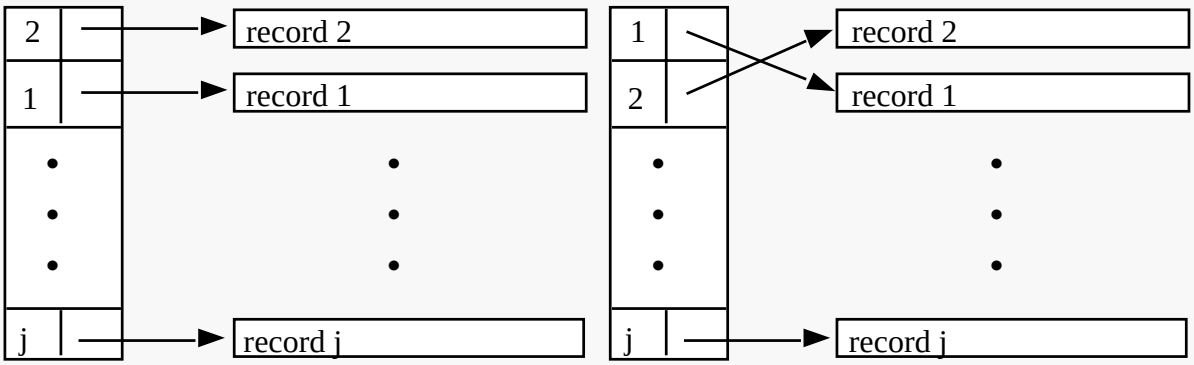
Minor Improvements

- All function calls should be replaced by inline code to avoid function overhead.
- Current partition bounds should be held in register variables.

Sorting Considerations

General

- With large data records, swap pointers instead of copying records:



before swapping first two keys

after swapping first two keys

There's a tradeoff here between storage cost and time. There's also a time versus time tradeoff. We're accepting the cost of additional pointer dereferences to avoid the cost of some data copying.

- Carefully investigate the average data arrangement in order to select the optimal sorting algorithm.
- No one algorithm works the best in all cases.

Better than n log(n)?

Special Techniques for Special Cases

- If sort key (member) consists of consecutive (unique) N integers they can be easily mapped onto the range 0 .. N-1 & sorted.
- If the N elements are initially in array A, then:

```
Item A[], B[];

for (int i = 0; i < N; i++ )
     B[A[i].GetKey() % N] = A[i];
```

  takes O(n) time.
- Requires exactly 1 record with each key value!
- Of course, this is a very special circumstance…
- Special case of Bin Sorting. (If integers are not consecutive, but within a reasonable range, used bit flags can be used to denote empty array slots.)

BinSort

BinSort is a somewhat more general O(N) sort technique.

Assume we need to sort an array of integers in the range 0-99:

34	16	83	76	40	72	38	80	89	87

Assume we have an array of 10 linked lists (bins) for storage.

First make a pass through the list of integers, and place each into the bin that matches its 1's digit.

Then, make a second pass, taking each bin in order, and place each integer into the bin that matches its 2's digit.

```
Bin
 0:   40   80
 1:
 2:   72
 3:   83
 4:   34
 5:
 6:   16   76
 7:   87
 8:   38
 9:   89
```

```
Bin
 0:
 1:   16
 2:
 3:   34   38
 4:   40
 5:
 6:
 7:   72   76
 8:   80   83   87   89
 9:
```

Now if you just read the bins, in order, the elements will appear in ascending order. Each pass takes O(N) work, and the number of passes is just the number of digits in the largest integer in the original list.

That beats QuickSort, but only in a somewhat special case.

BinSort Implementation

```
void BinSort(int Data[], int numData, LinkList Bin[]) {

   // First pass:
   for (int Idx = 0; Idx < numData; Idx++) {
      int Digit1 = Data[Idx] % 10;
      Bin[Digit1].gotoTail();
      Bin[Digit1].Insert(Item(Data[Idx]));//append to end
   }
   // Second pass:
   LinkList Bin2[NumBins];
   for (Idx = 0; Idx < 10; Idx++) {
      Bin[Idx].gotoHead();
      while (Bin[Idx].inList()) {
         int currValue =
                Bin[Idx].getCurrentData().getValue();
         int Digit2 = (currValue / 10) % 10;
         Bin2[Digit2].gotoTail(); //append to end
         Bin2[Digit2].Insert(Item(currValue));
         Bin[Idx].Advance();
      }
   }
   LinearizeBins(Bin2, Data);
}
```

Note the modular reuse here of the linked list code covered earlier.

```
void LinearizeBins(LinkList Bin[], int Target[]) {

   int Tidx = 0;

   for (int Idx = 0; Idx < 10; Idx++) {
      Bin[Idx].gotoHead();
      while (Bin[Idx].inList()) {
         Target[Tidx] =
                 Bin[Idx].getCurrentData().getValue();
         Tidx++;
         Bin[Idx].Advance();
      }
   }
}
```