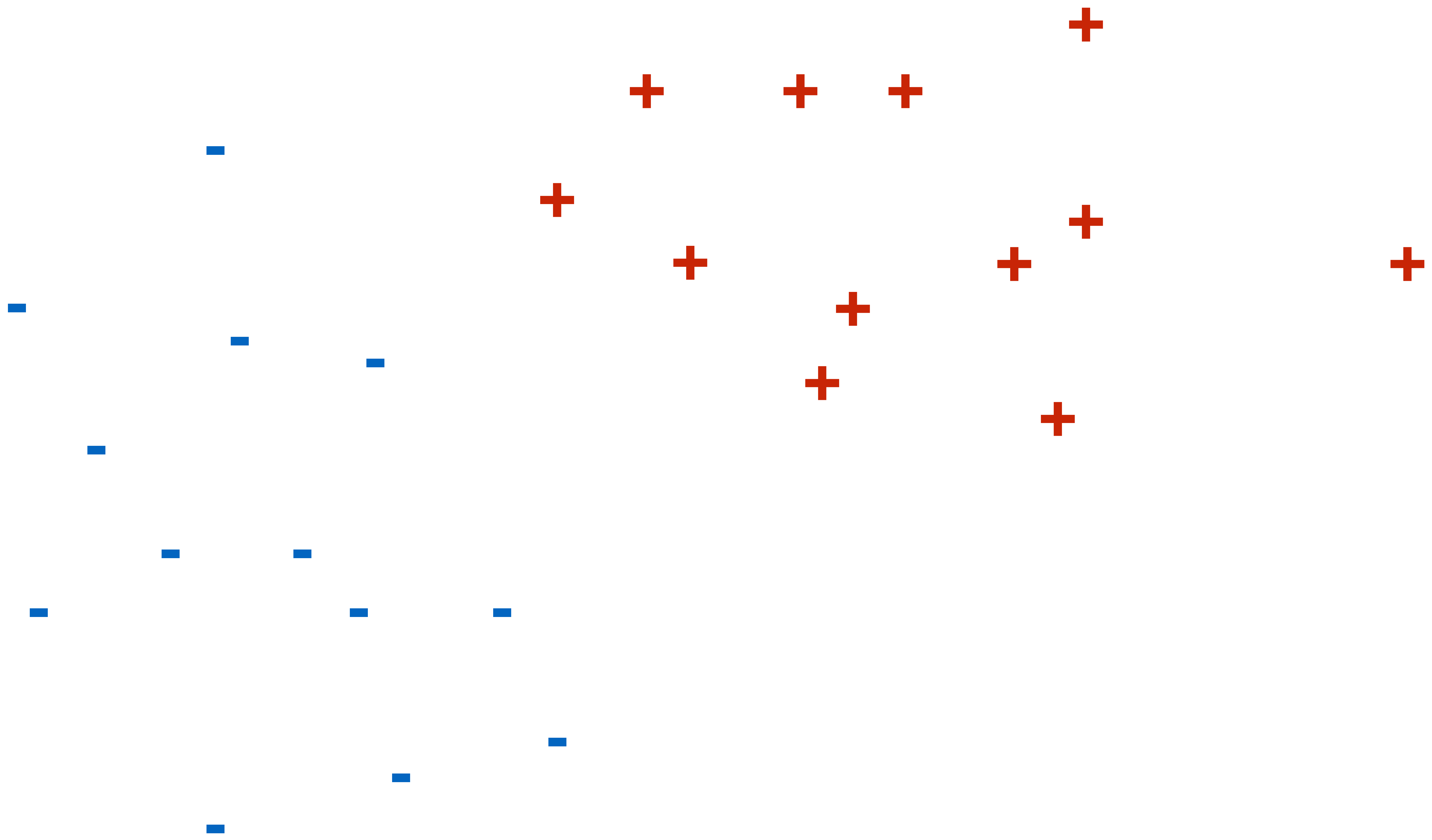
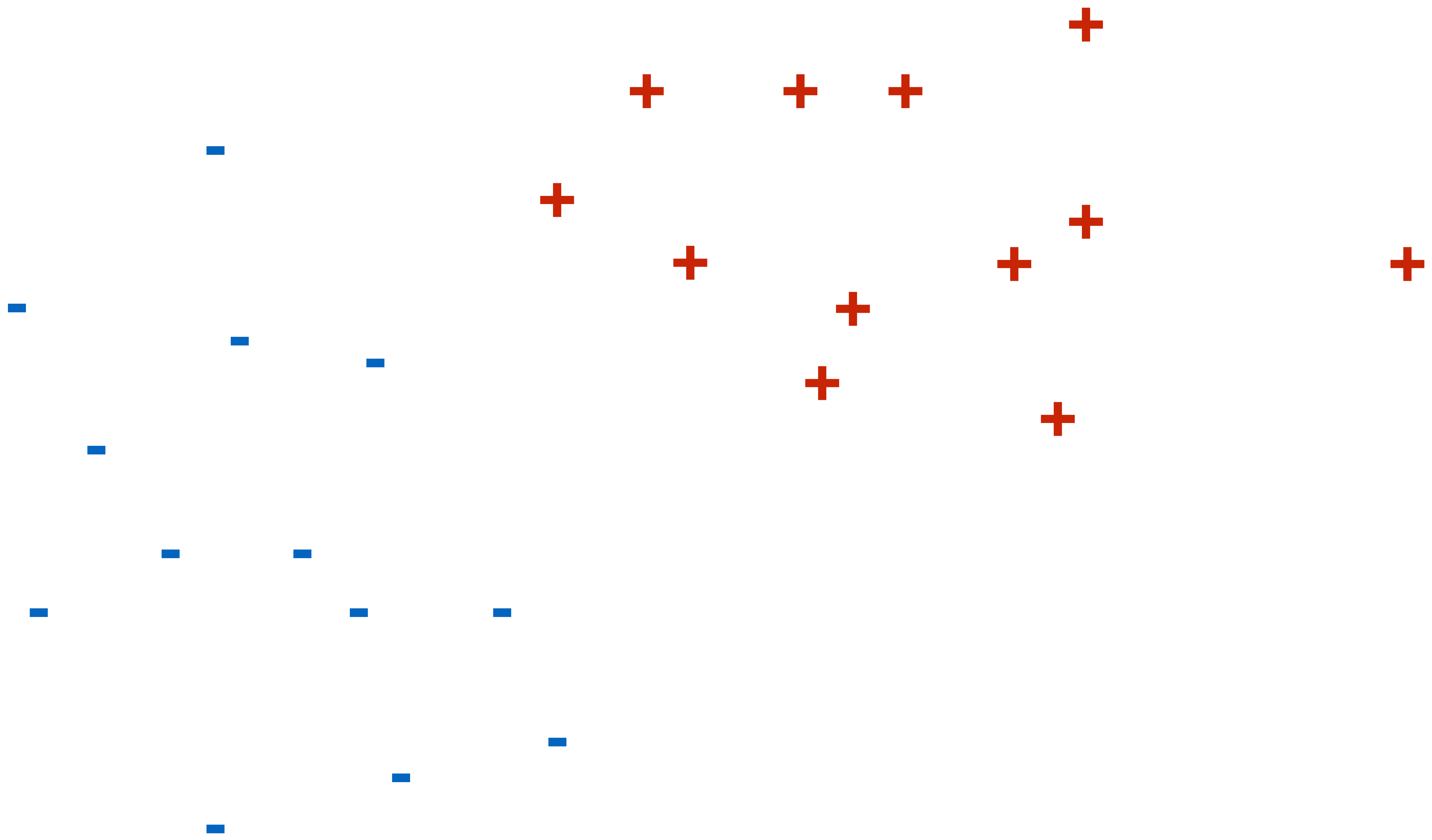
# Support Vector Machines

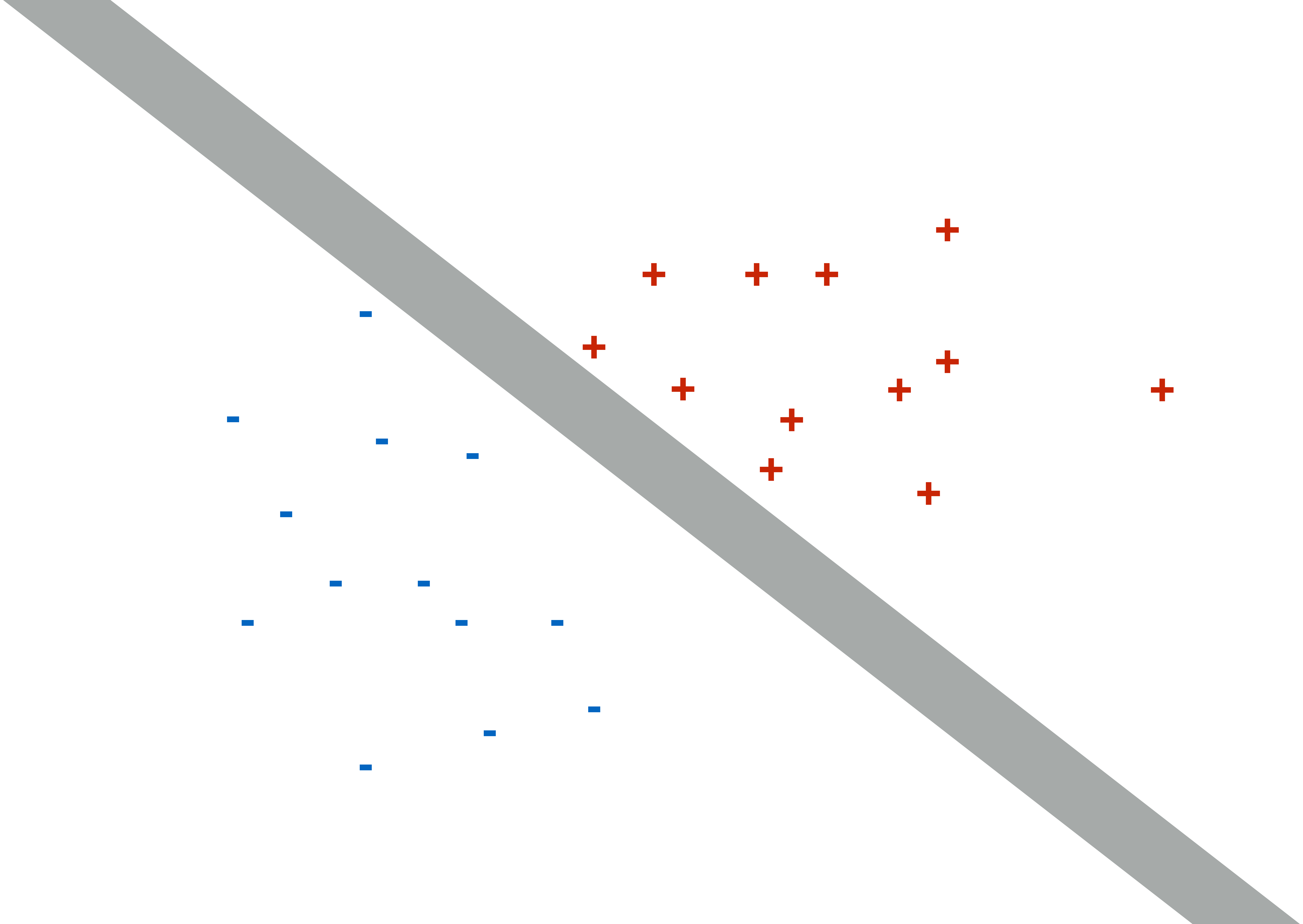
Machine Learning
CS5824/ECE5424
Bert Huang
Virginia Tech

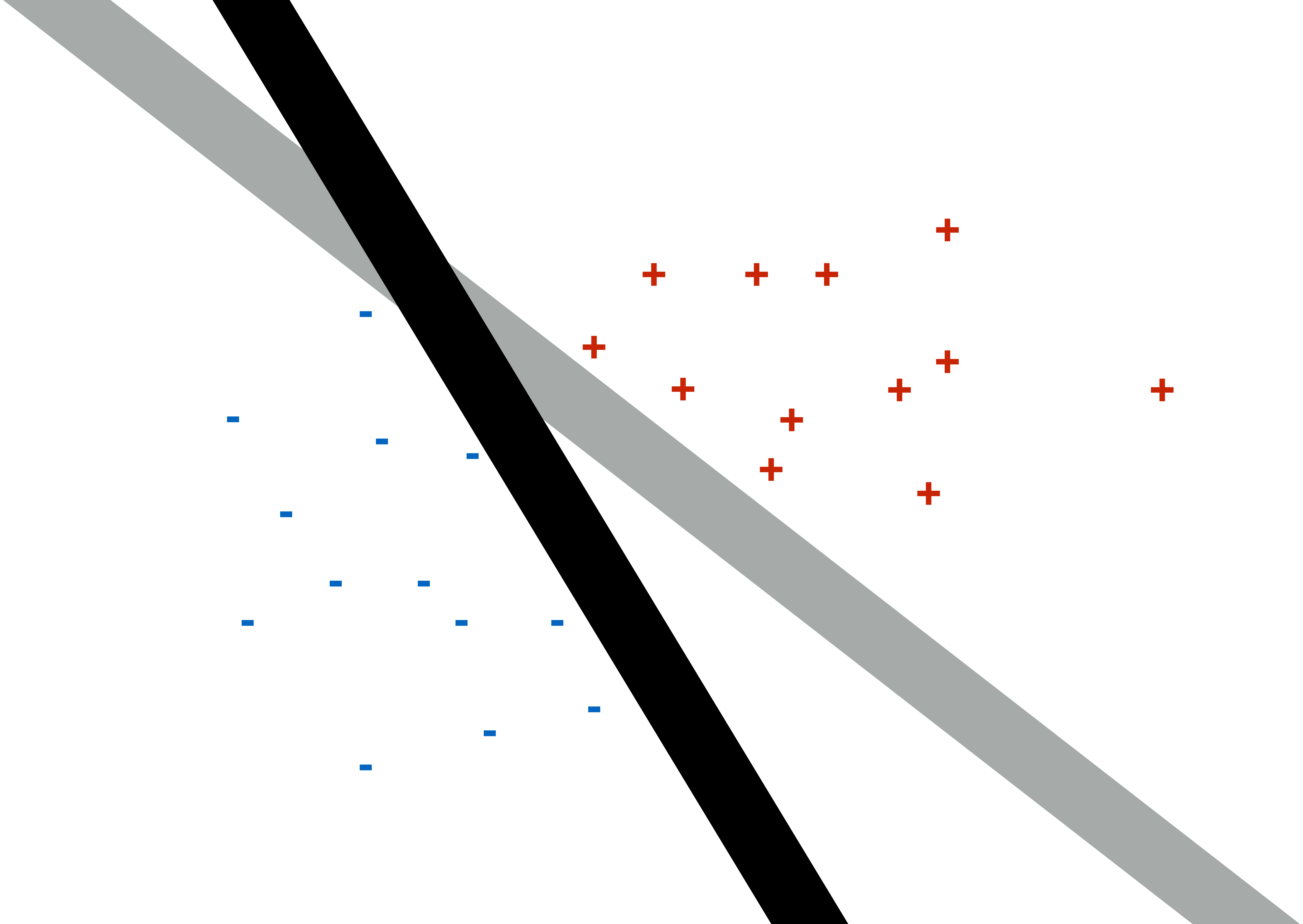
# Outline

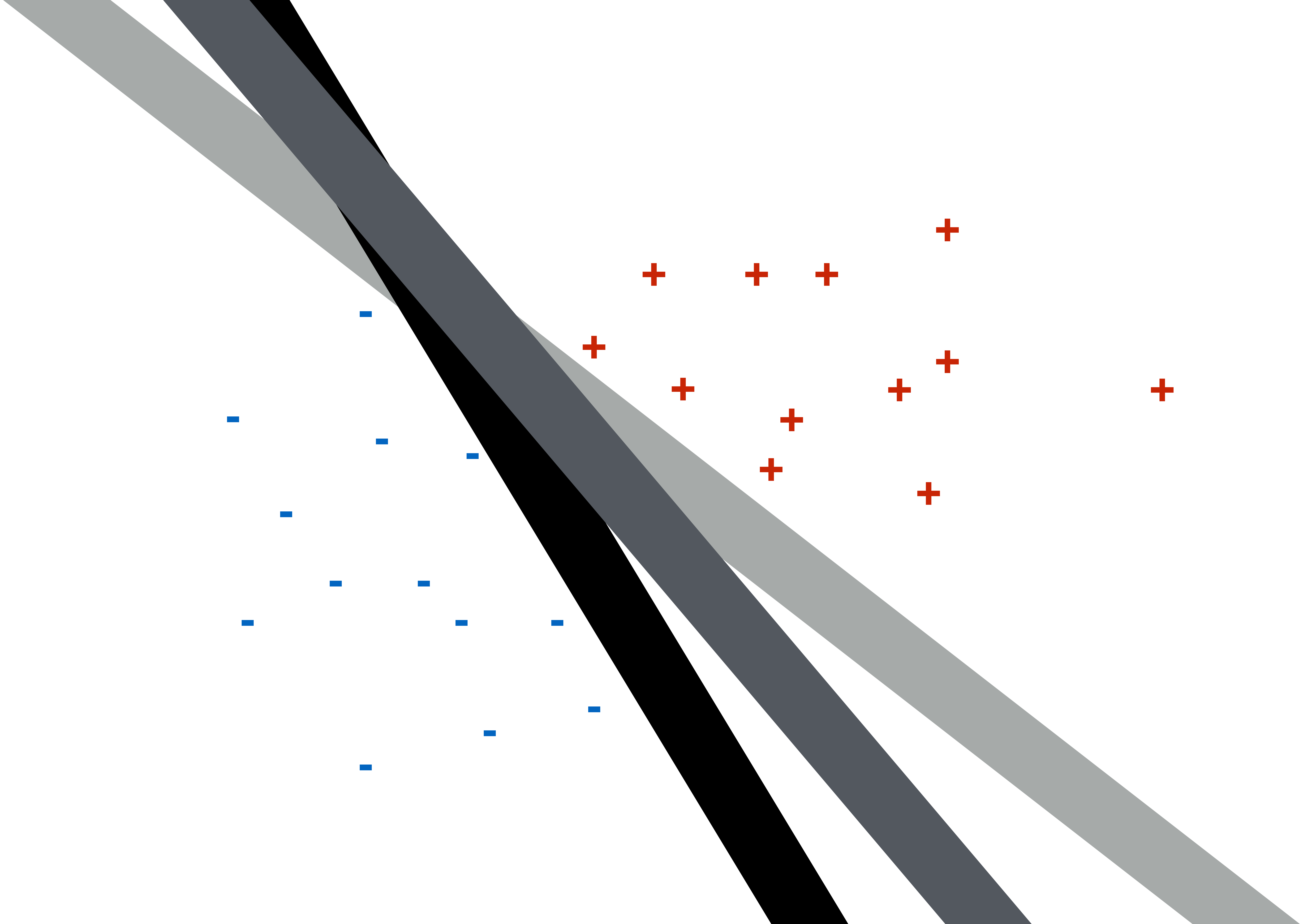
- Large-margin and model complexity
- Formalizing large margin
- Quadratic program form
- Soft-margin
- Non-linearity

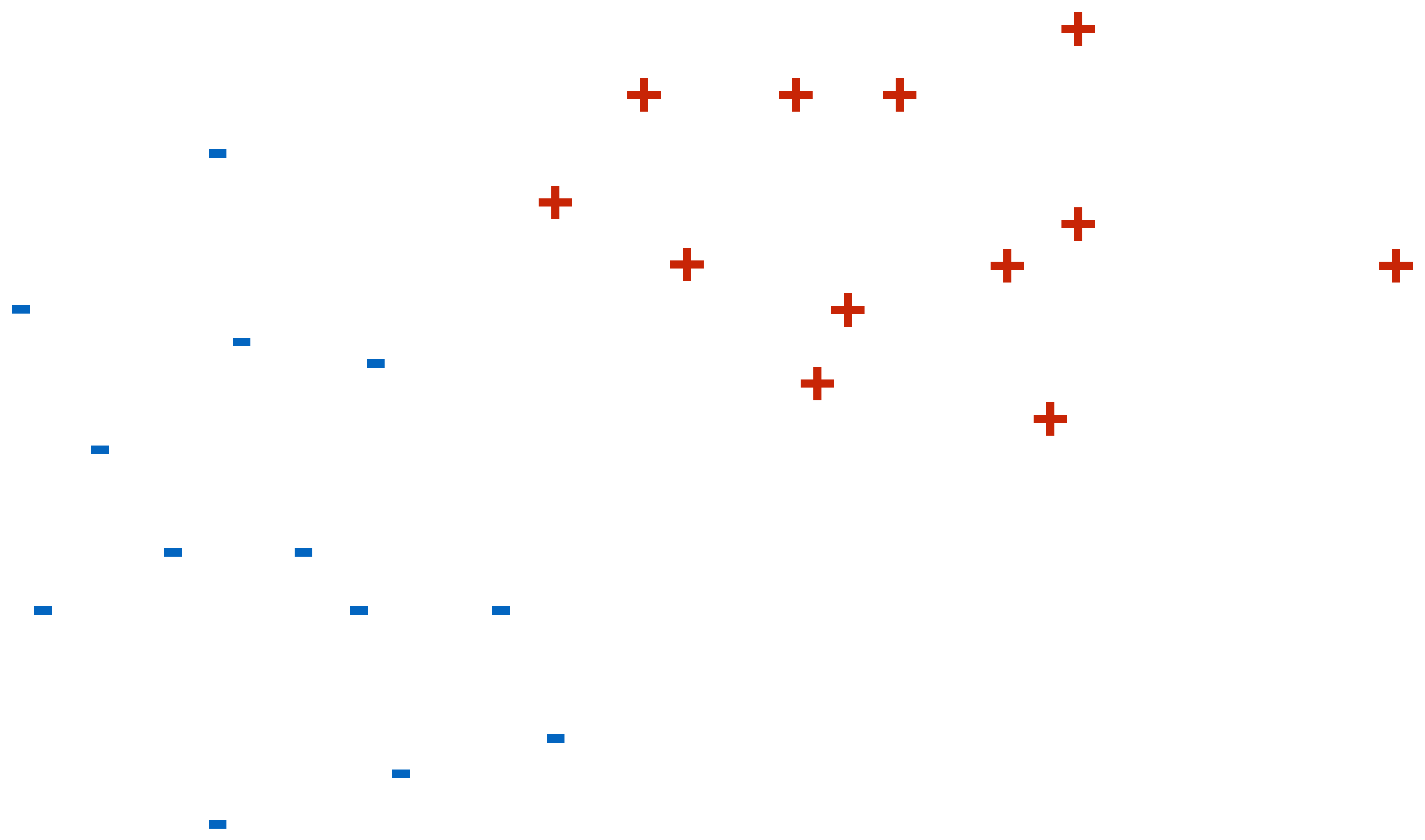


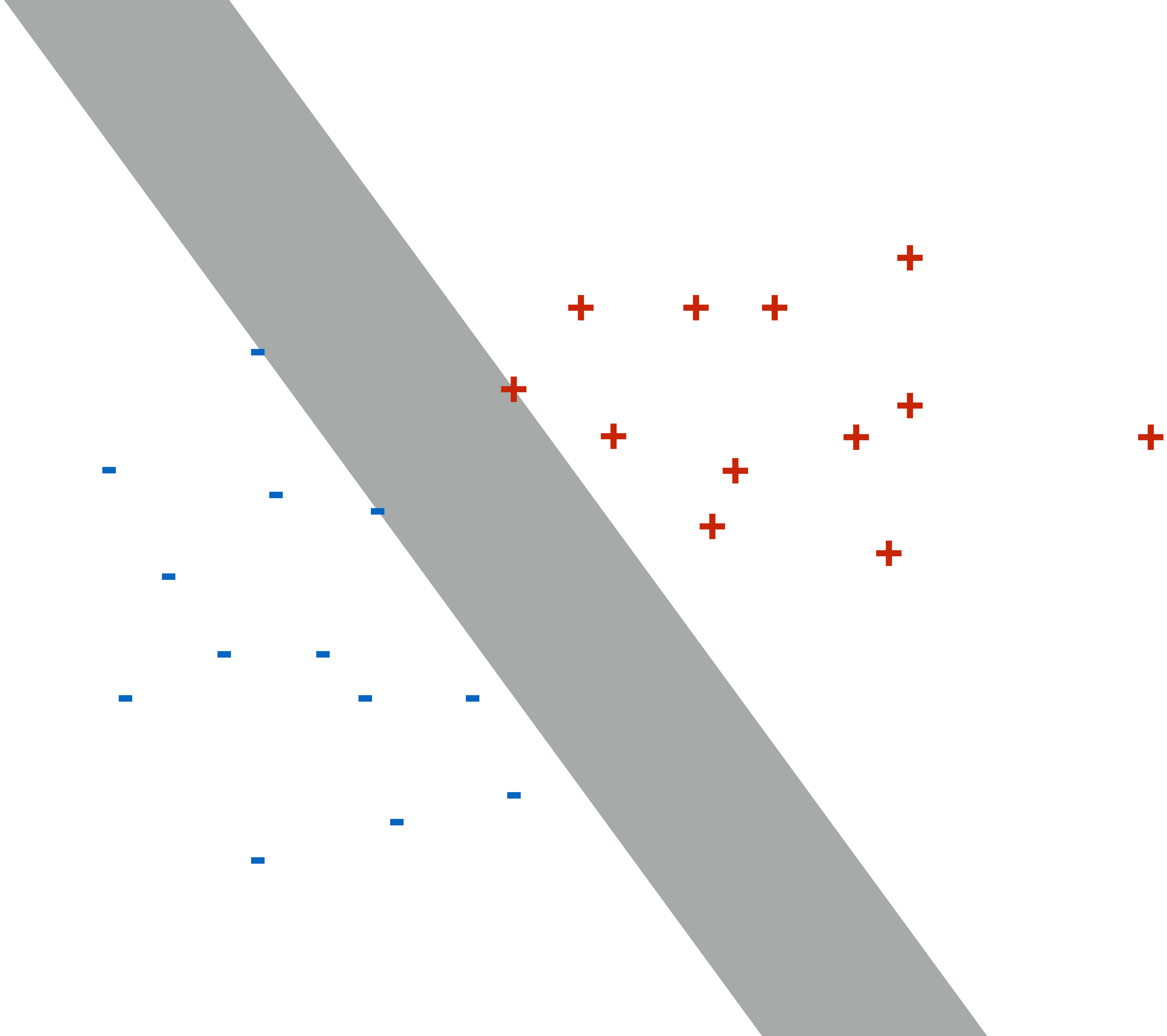


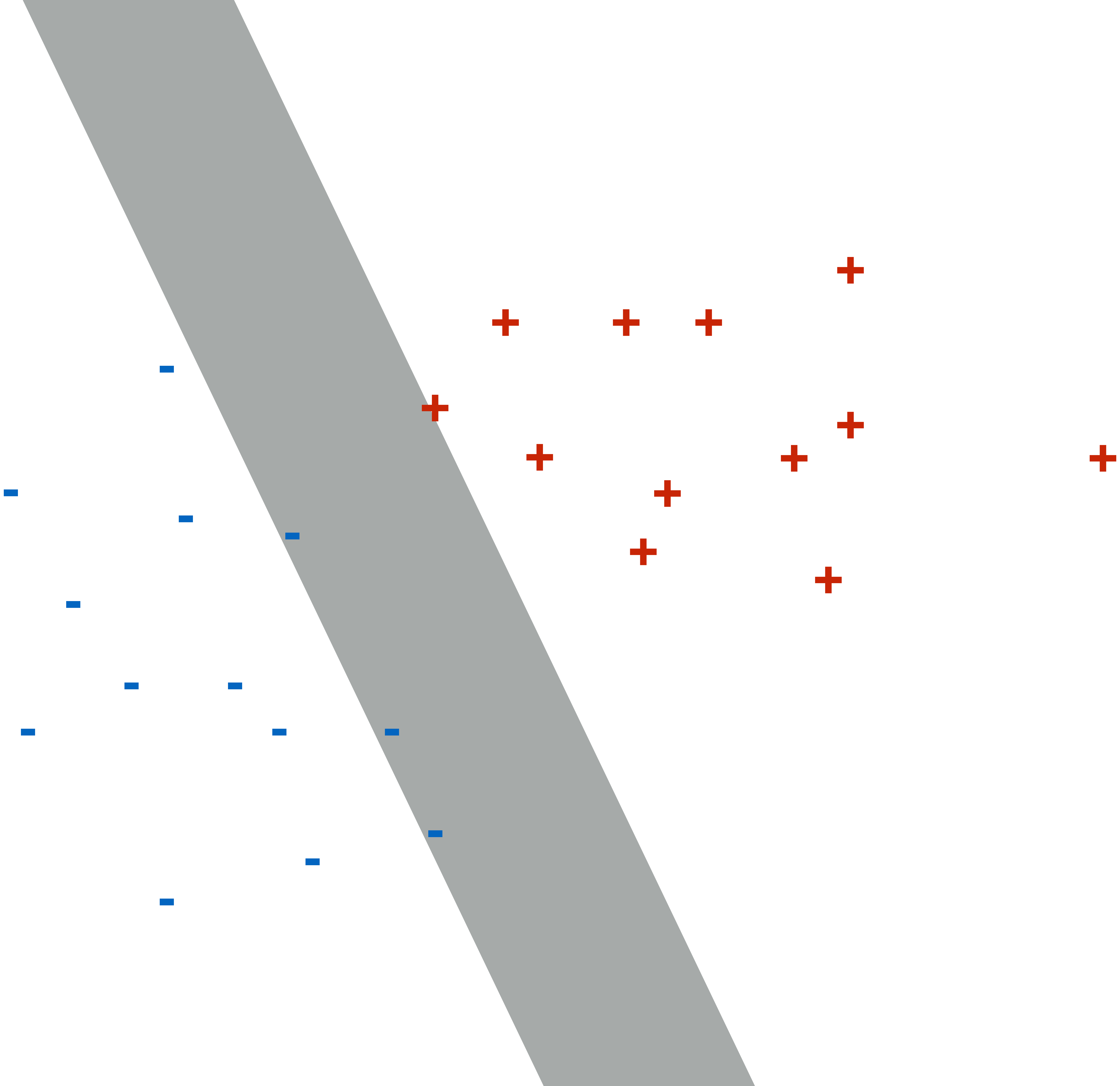


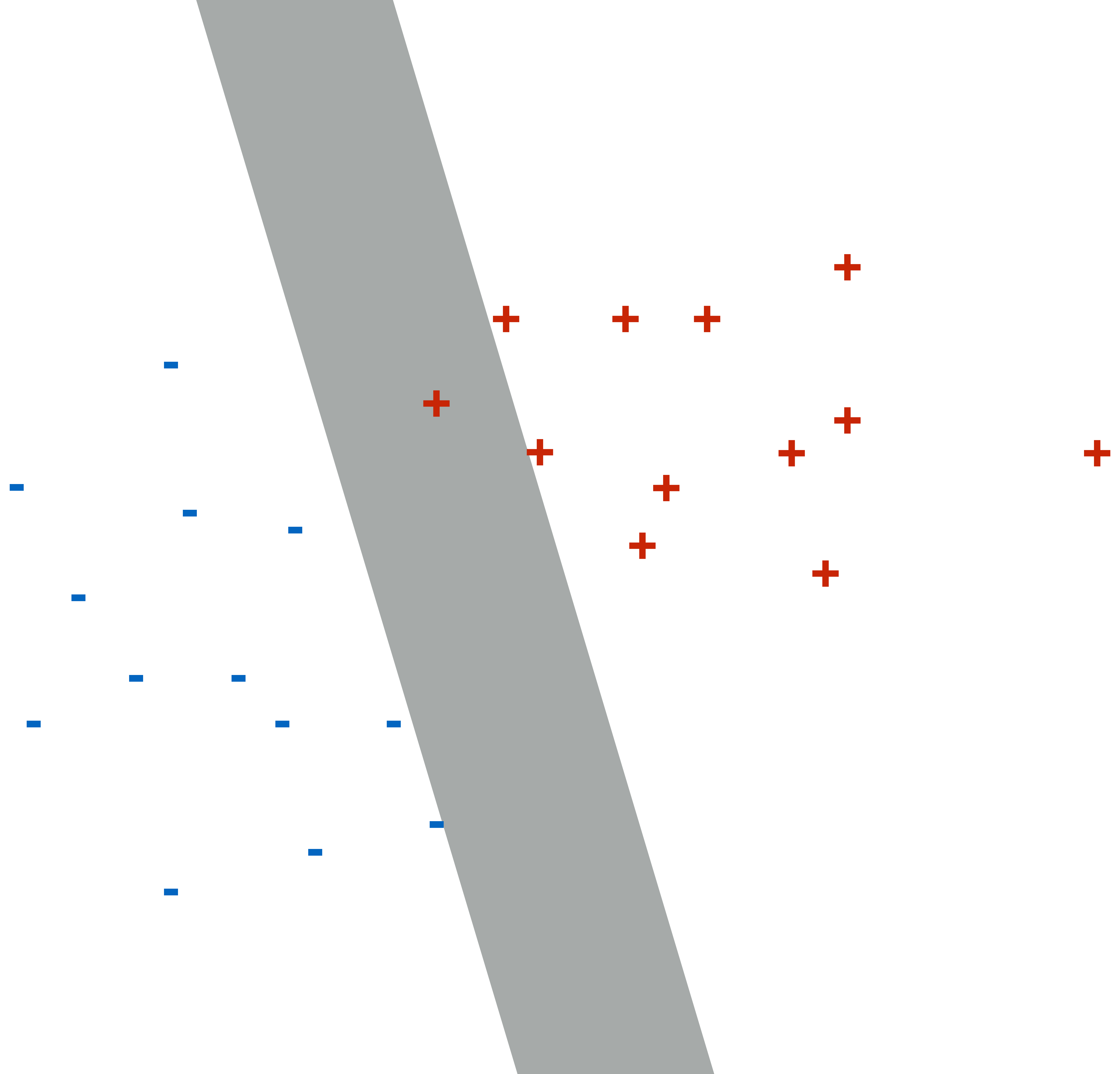


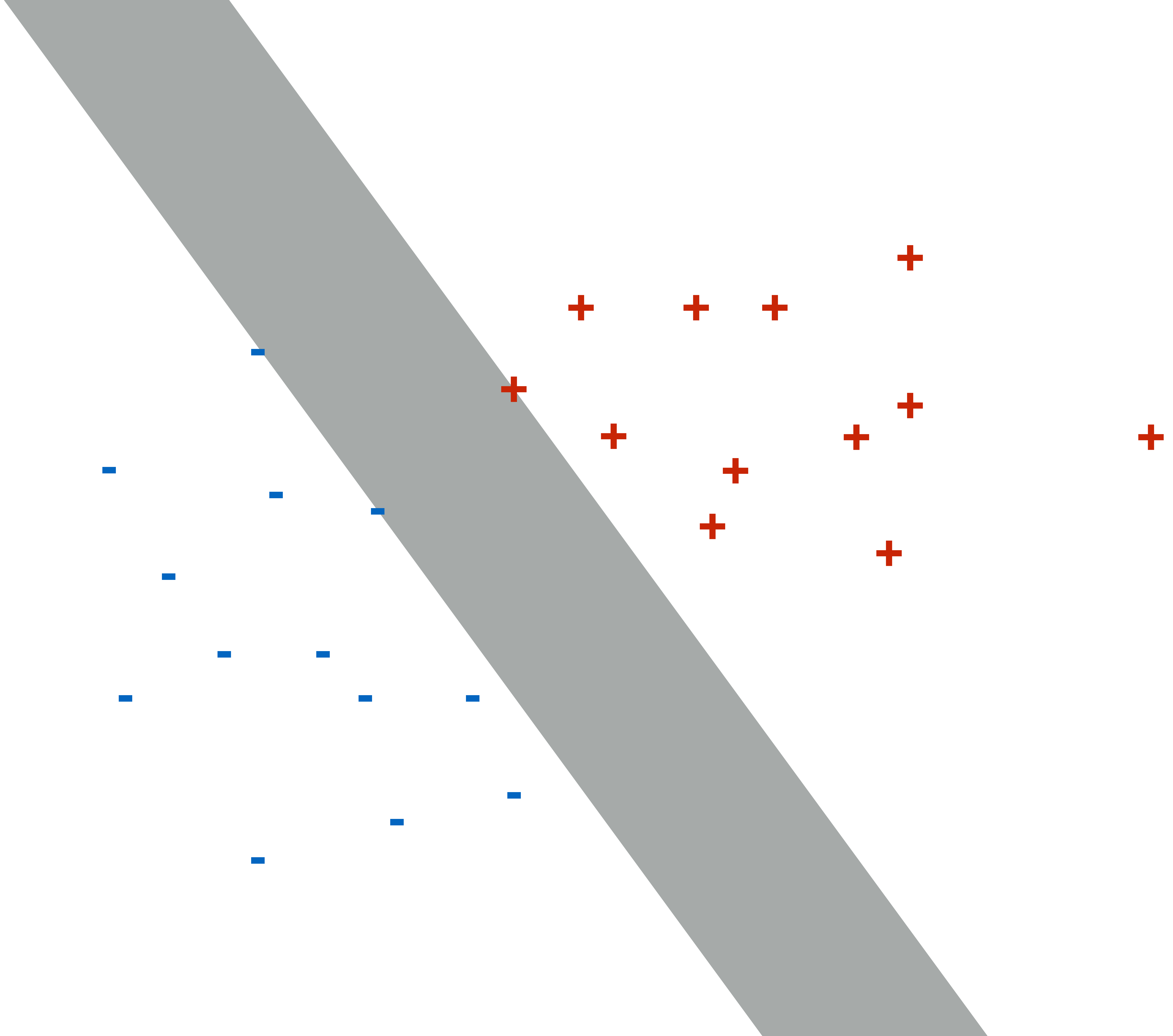


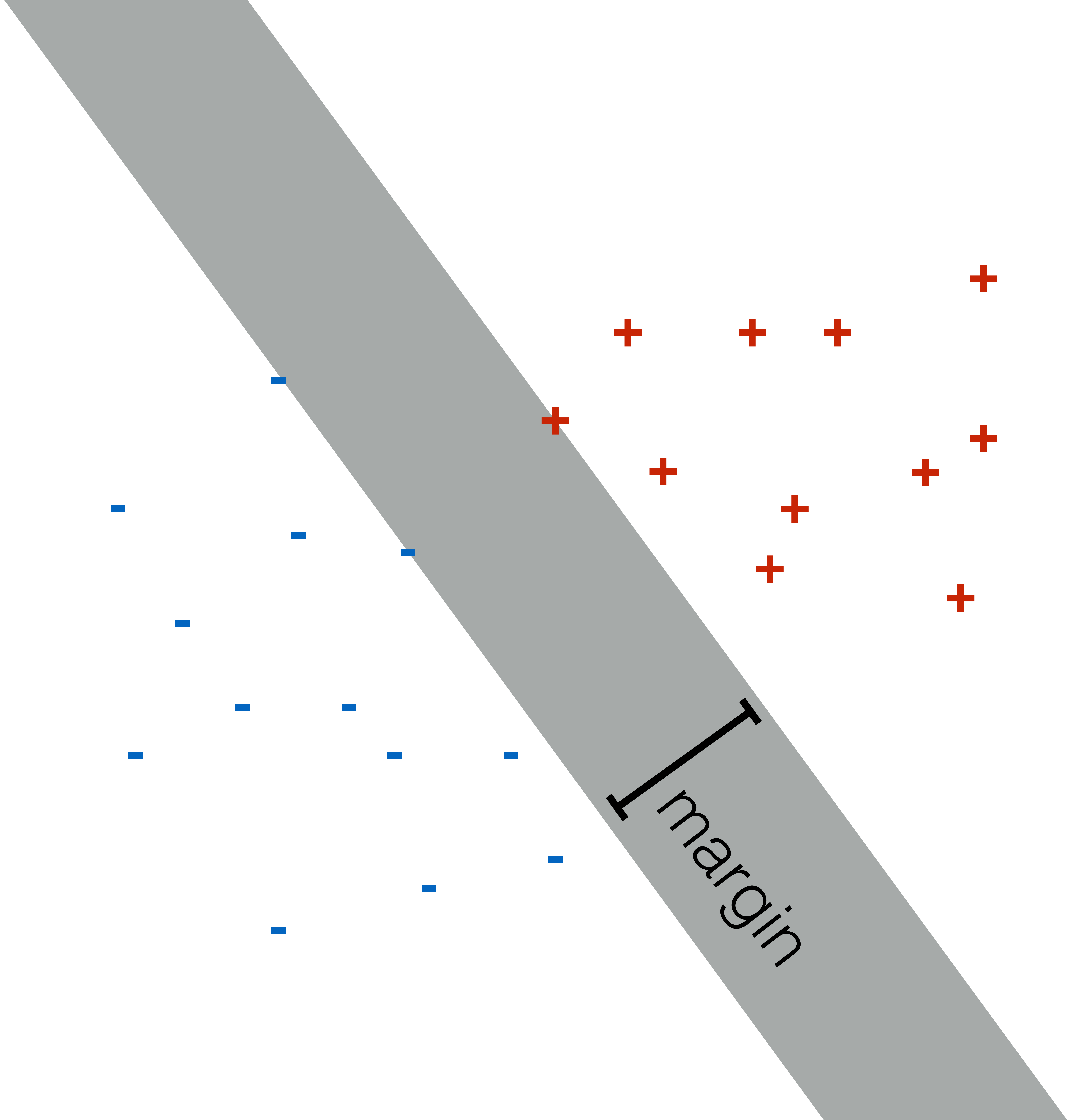








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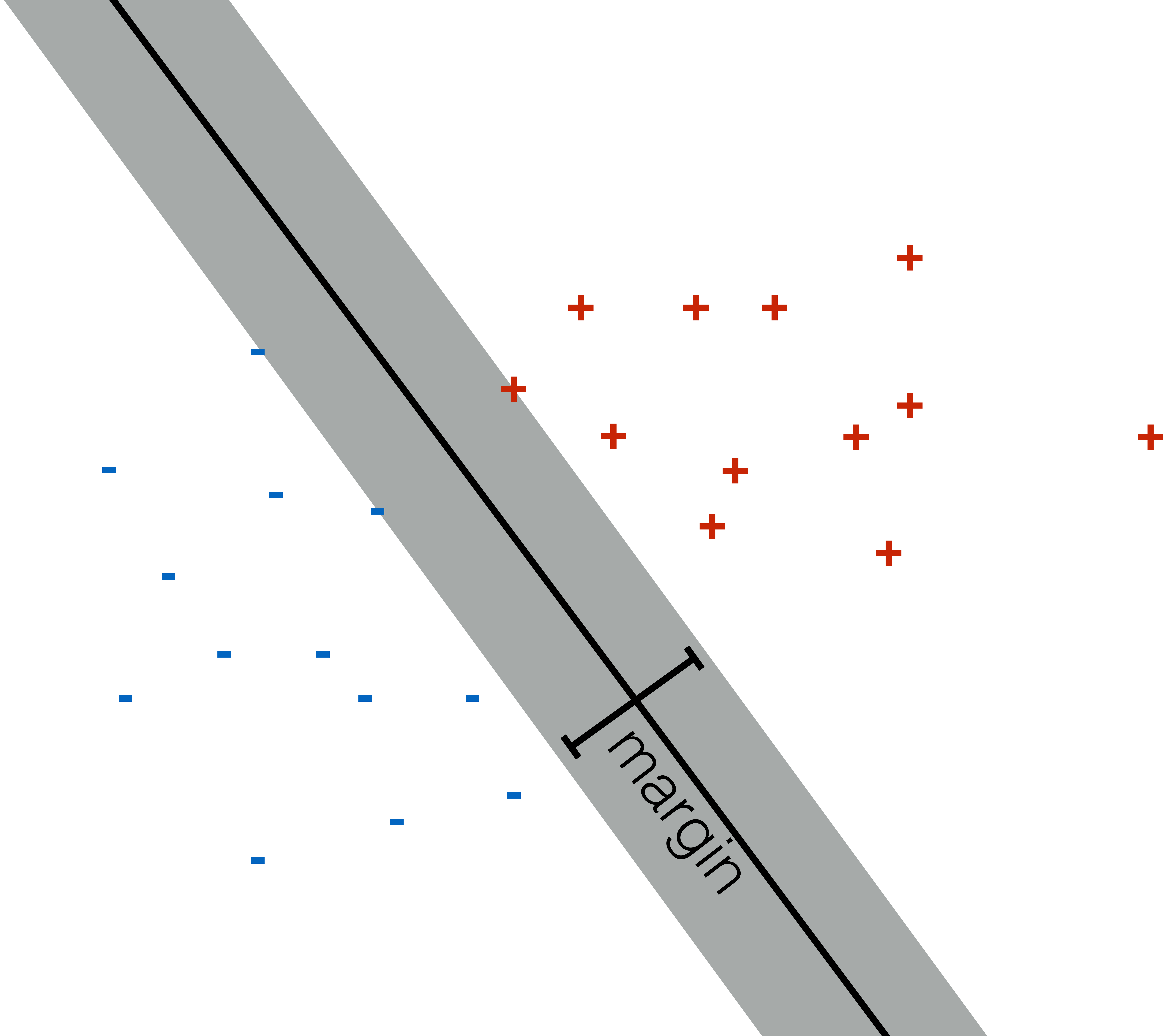
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# Quantifying the Margin

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$D = \{(x_1, y_1), \ldots, (x_n, y_n)\}$

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$$y_i(w^\top x_i + b) \geq 0$$

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$y_i(w^\top x_i + b) \geq 1 \quad \forall i \in \{1, \dots, n\}$

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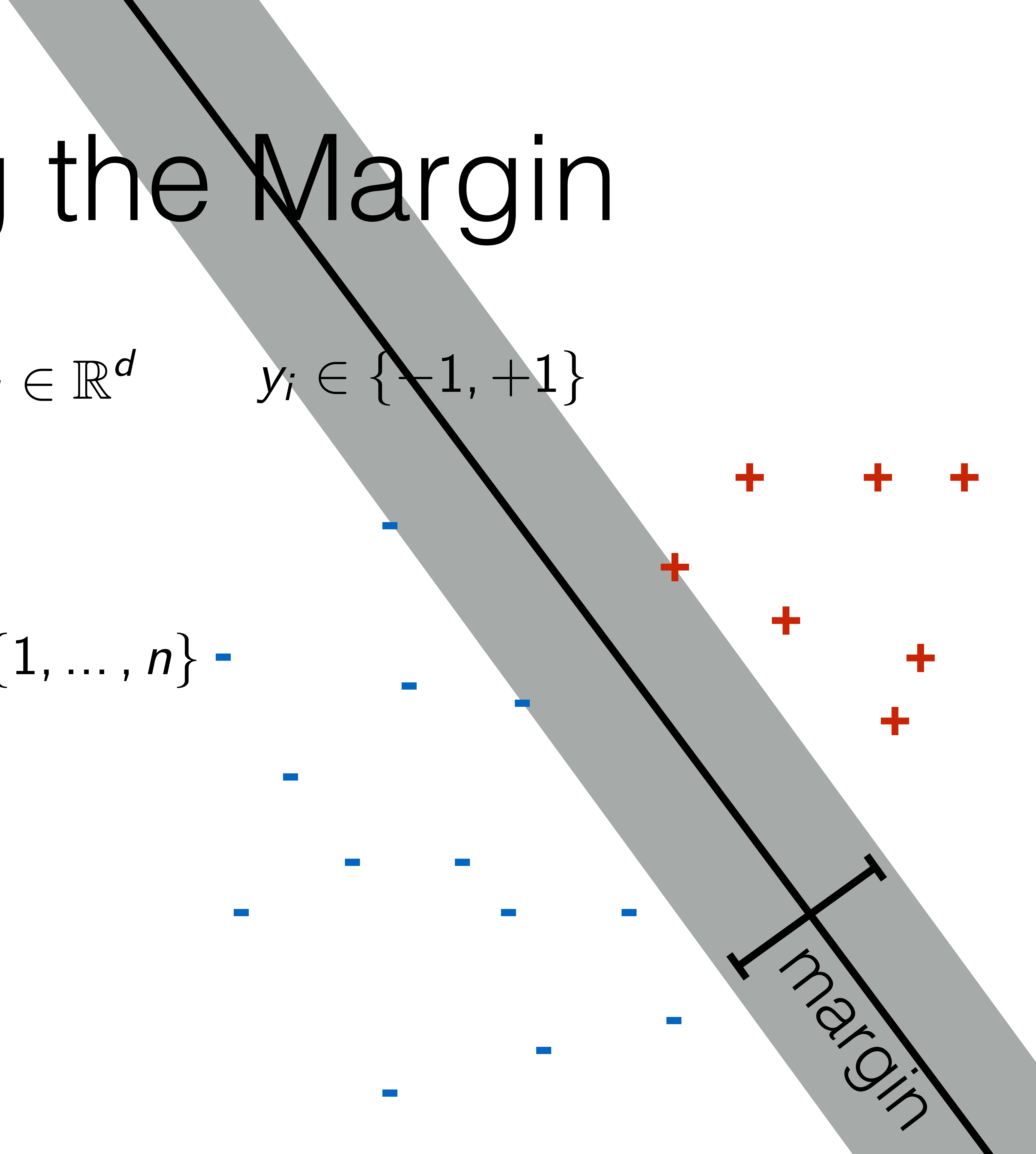


$w^\top x + b \geq 1$

$w^\top x + b \leq -1$
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$x_+$


$w^\top x + b \leq -1$
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$X_+$
$X_-$

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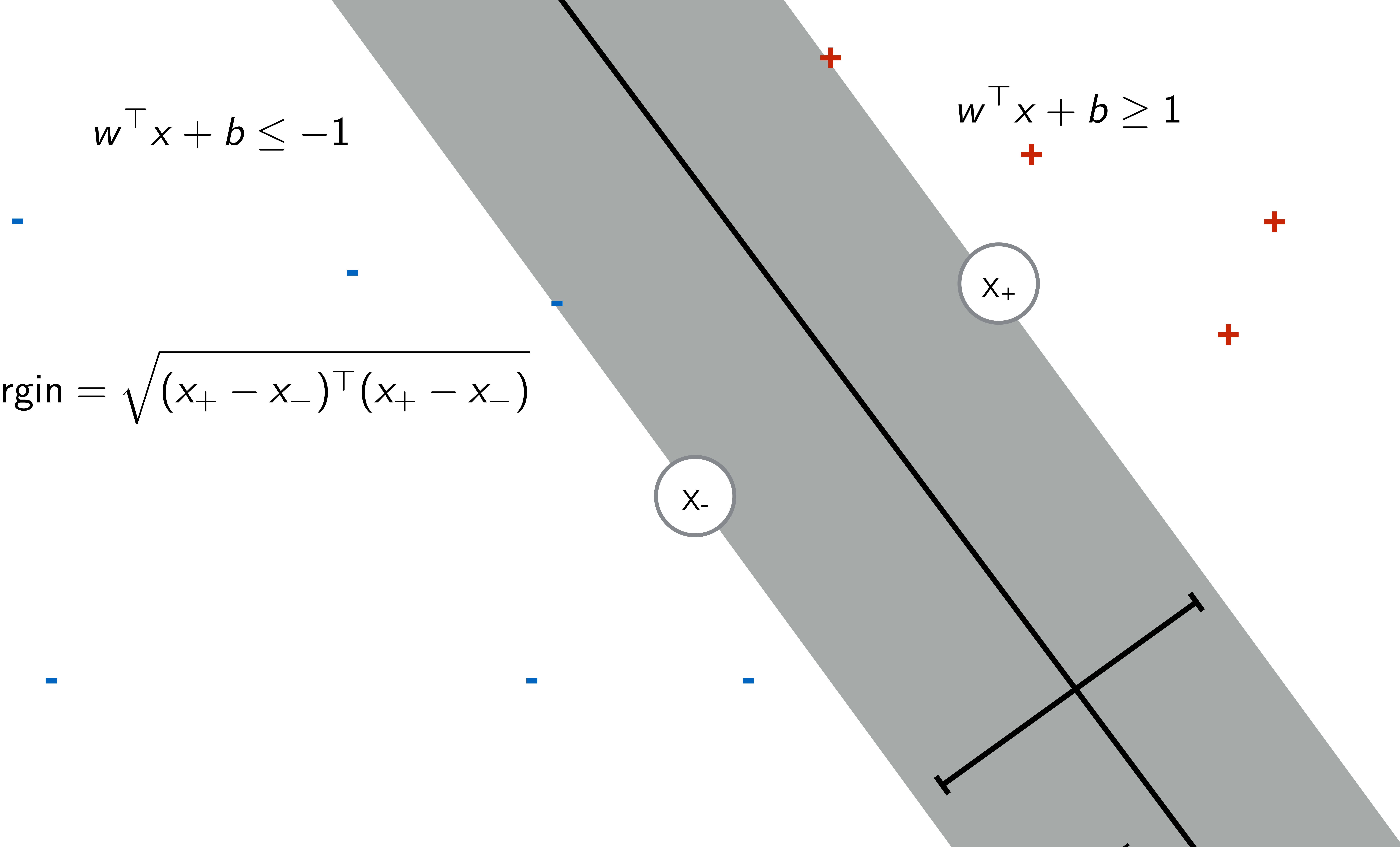
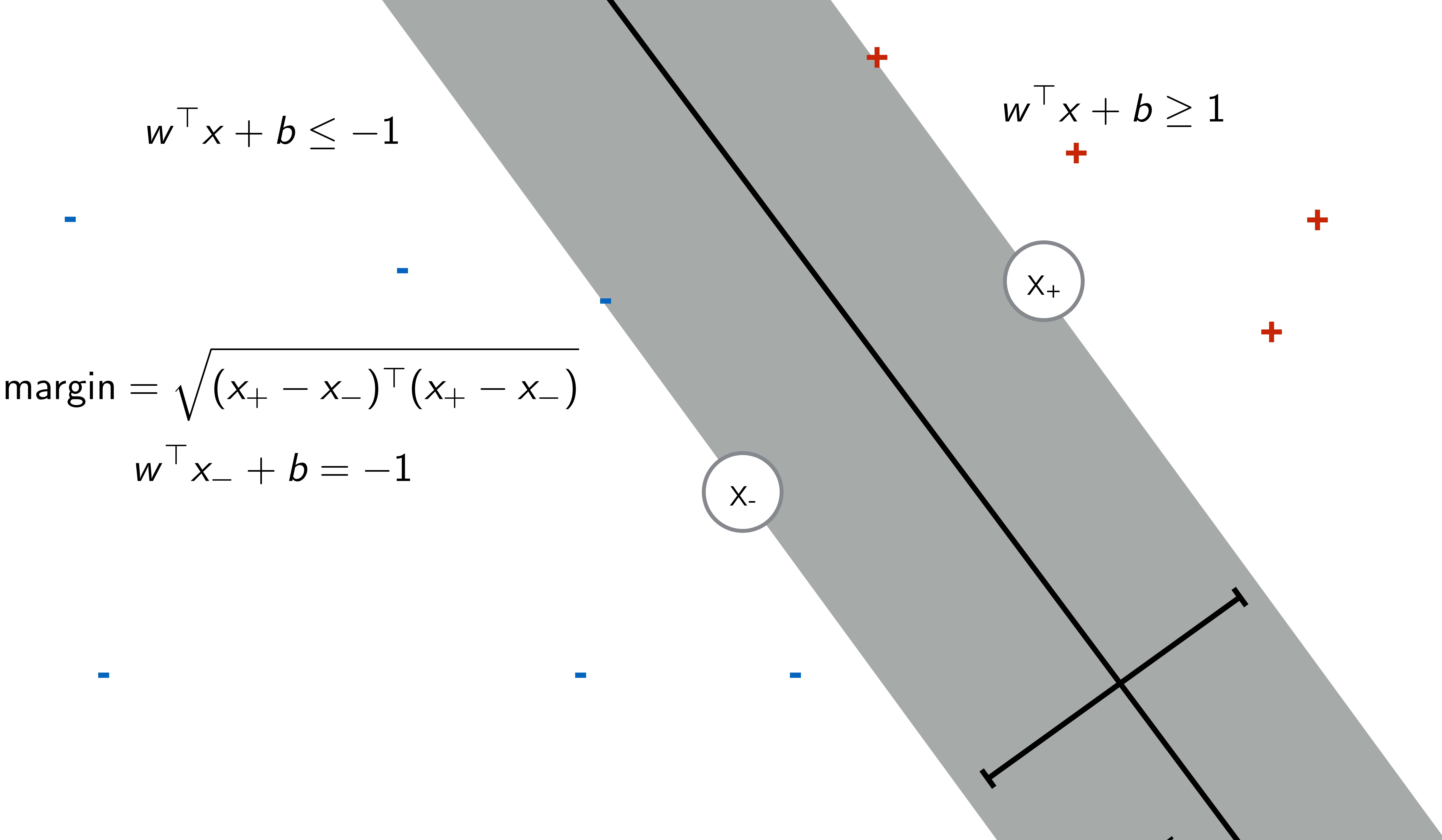
$$w^\top x + b \geq 1$$

$$\text{margin} = \sqrt{(x_+ - x_-)^\top (x_+ - x_-)}$$

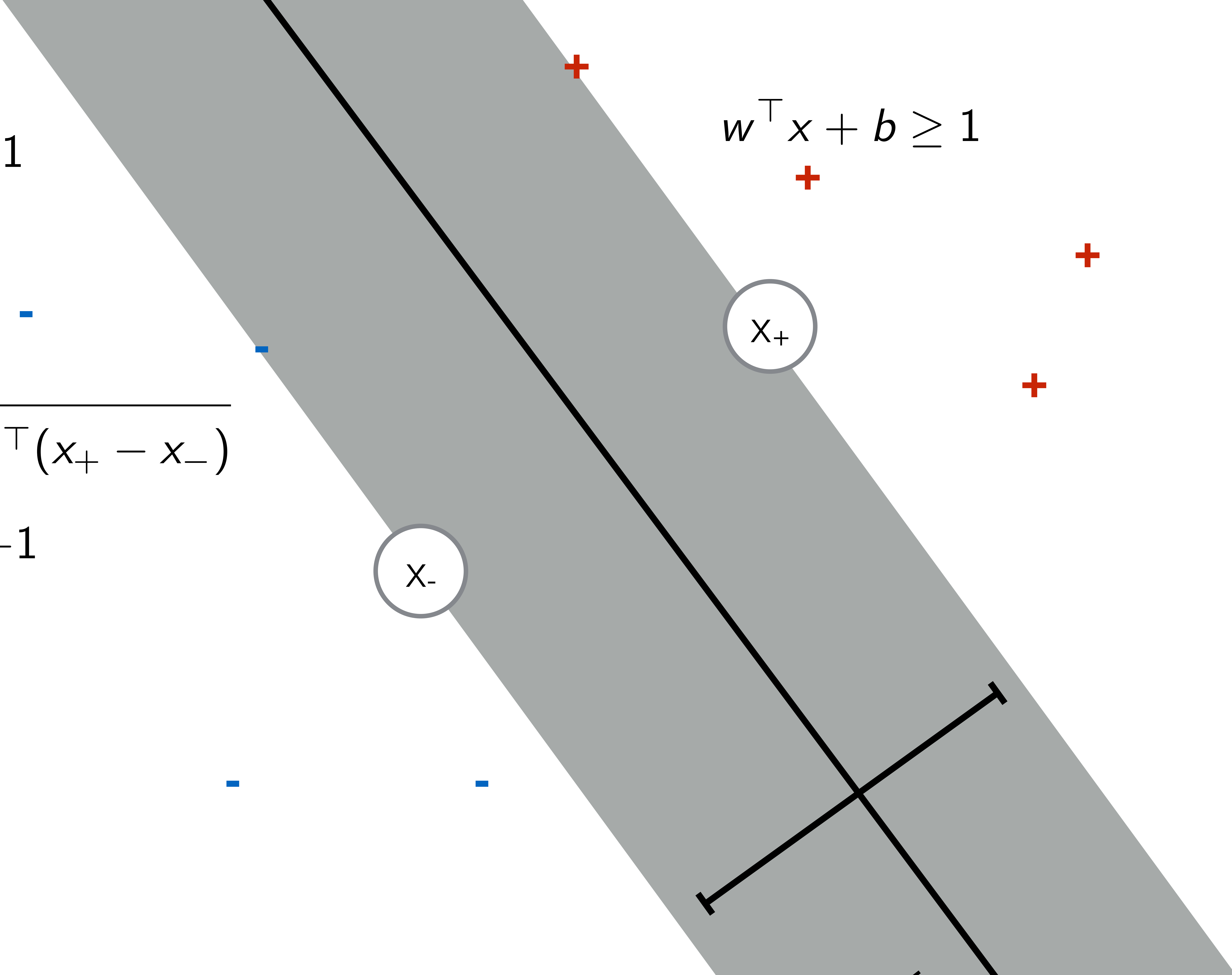




$w^\top x + b \leq -1$
$w^\top x + b \geq 1$
X+
X-
$\text{margin} = \sqrt{(x_+ - x_-)^\top (x_+ - x_-)}$
$w^\top x_- + b = -1$

$$w^\top x + b \leq -1$$

$$w^\top x + b \geq 1$$

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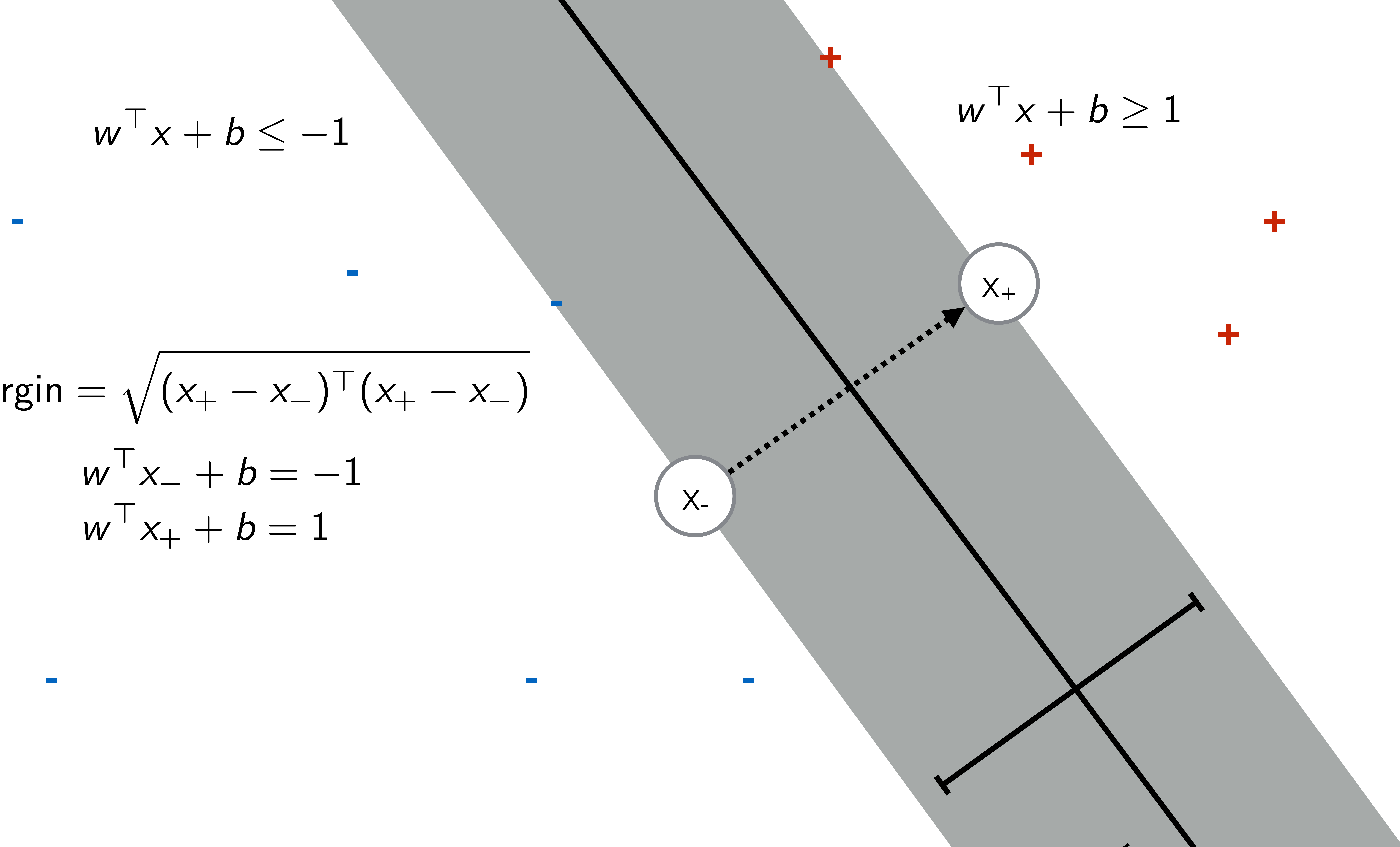


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$$w^\top x_- + b = -1$$

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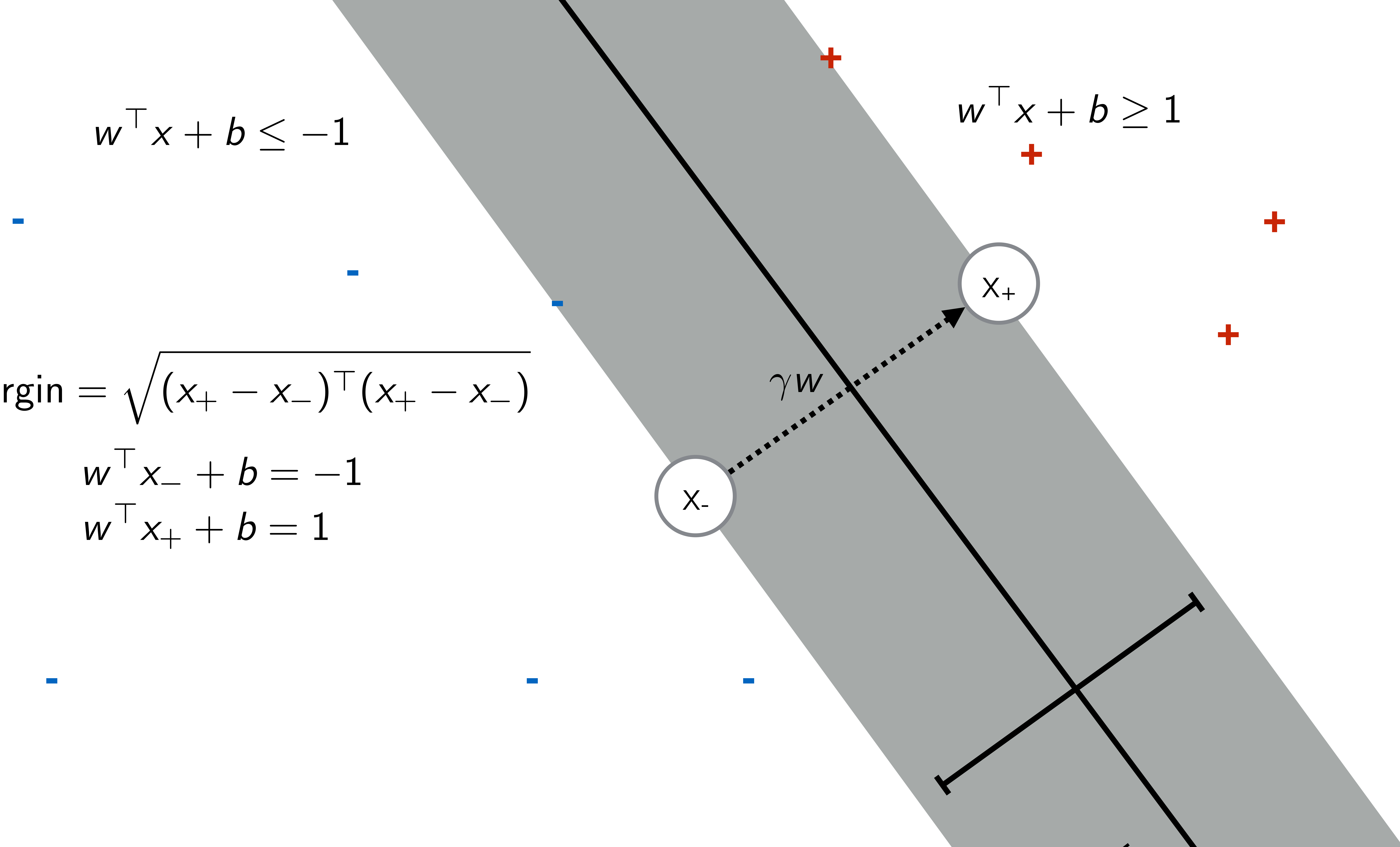

$$w^\top x + b \le -1$$

$$\text{margin} = \sqrt{(x_+ - x_-)^\top (x_+ - x_-)}$$

$$w^\top x_- + b = -1$$

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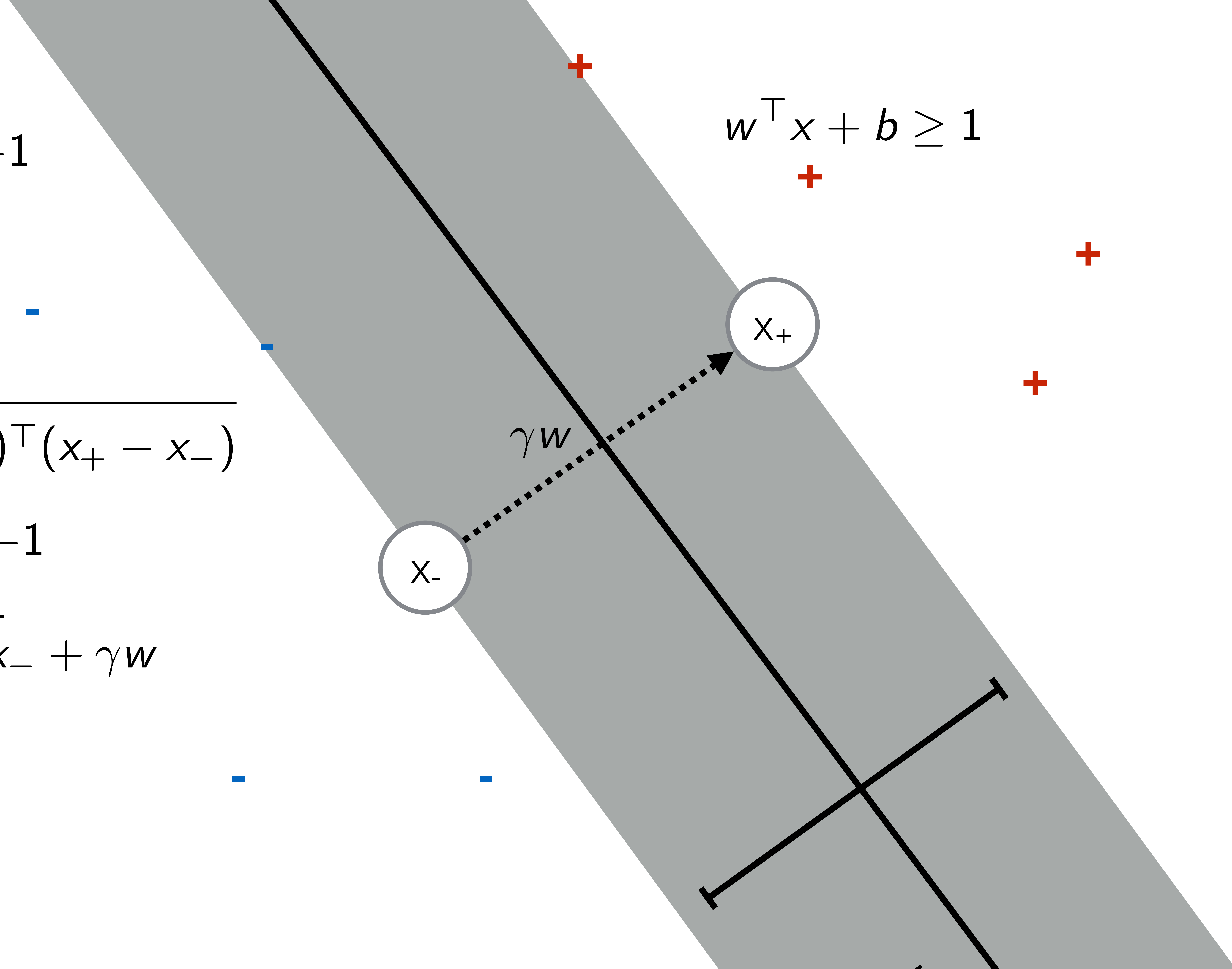
$$x_+ = x_- + \gamma w$$



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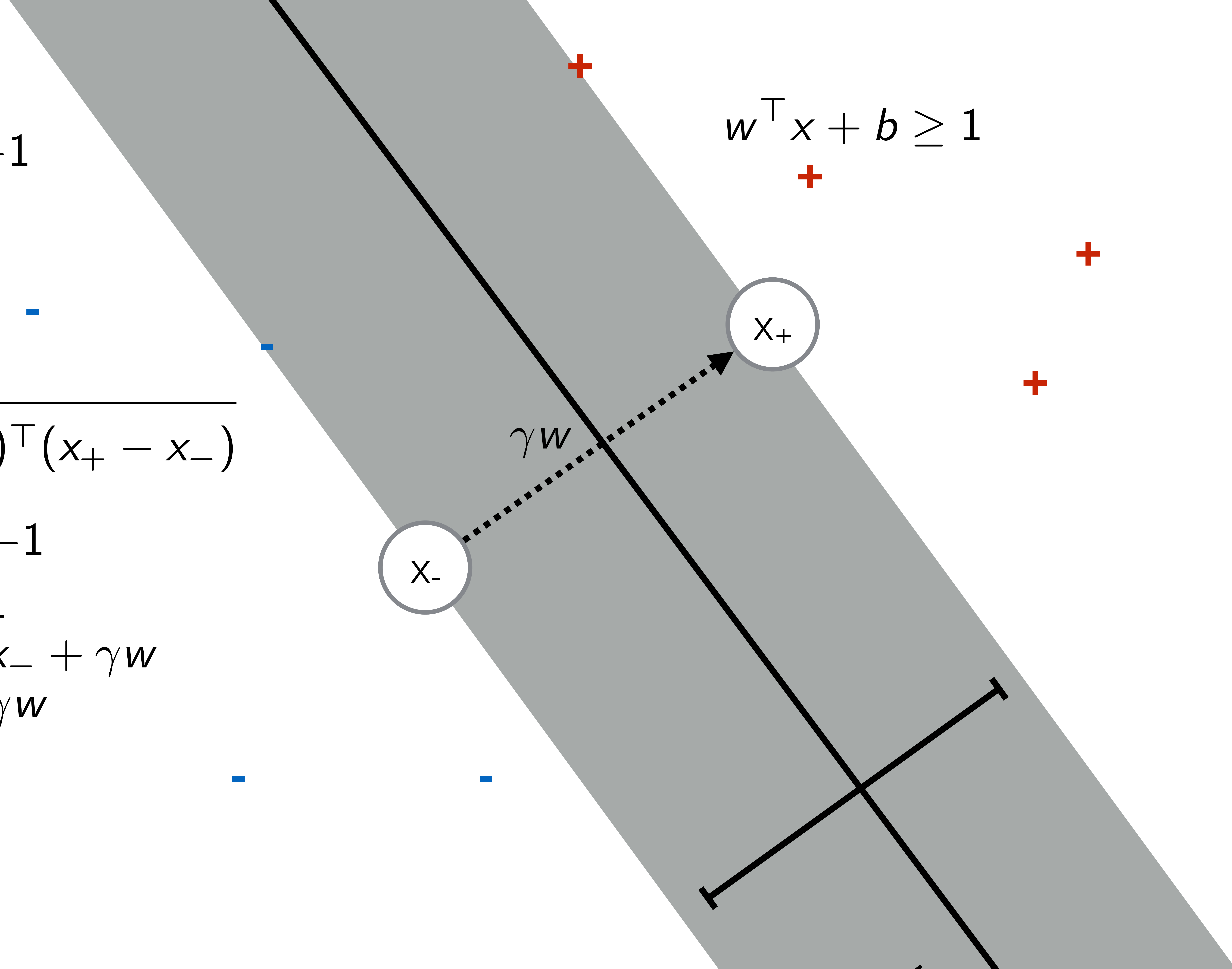

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$$\gamma = \frac{2}{w^\top w}$$

$$\text{margin} = \sqrt{\frac{4w^\top w}{w^\top w \times w^\top w}} = \frac{2}{\sqrt{w^\top w}}$$

# Large-Margin Linear Classification

$$\max_{w \in \mathbb{R}^d} \quad \frac{2}{\sqrt{w^\top w}}$$

$$\text{s.t.} \quad y_i(w^\top x_i + b) \geq 1 \quad \forall i \in \{1, \dots, n\}$$

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# Large-Margin Linear Classification

$$\min_{w \in \mathbb{R}^d} \quad \frac{1}{2} w^\top w$$

$$\text{s.t.} \quad y_i(w^\top x_i + b) \geq 1 \qquad \forall i \in \{1, \dots, n\}$$

# Quadratic Programming

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$$\min_{x} \quad \frac{1}{2} x^\top H x + f^\top x$$

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$$\min_{x} \quad \frac{1}{2}x^\top H x + f^\top x$$ quadratic objective

$$\text{s.t.} \quad A_{\text{ineq}} x \leq b_{\text{ineq}}$$ linear inequality constraints

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linear inequality constraints

$$A_{\text{eq}} x = b_{\text{eq}}$$

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$$A_{\text{eq}}x = b_{\text{eq}}$$ linear equality constraints

$$\min_{w \in \mathbb{R}^d} \quad \frac{1}{2} w^\top w$$

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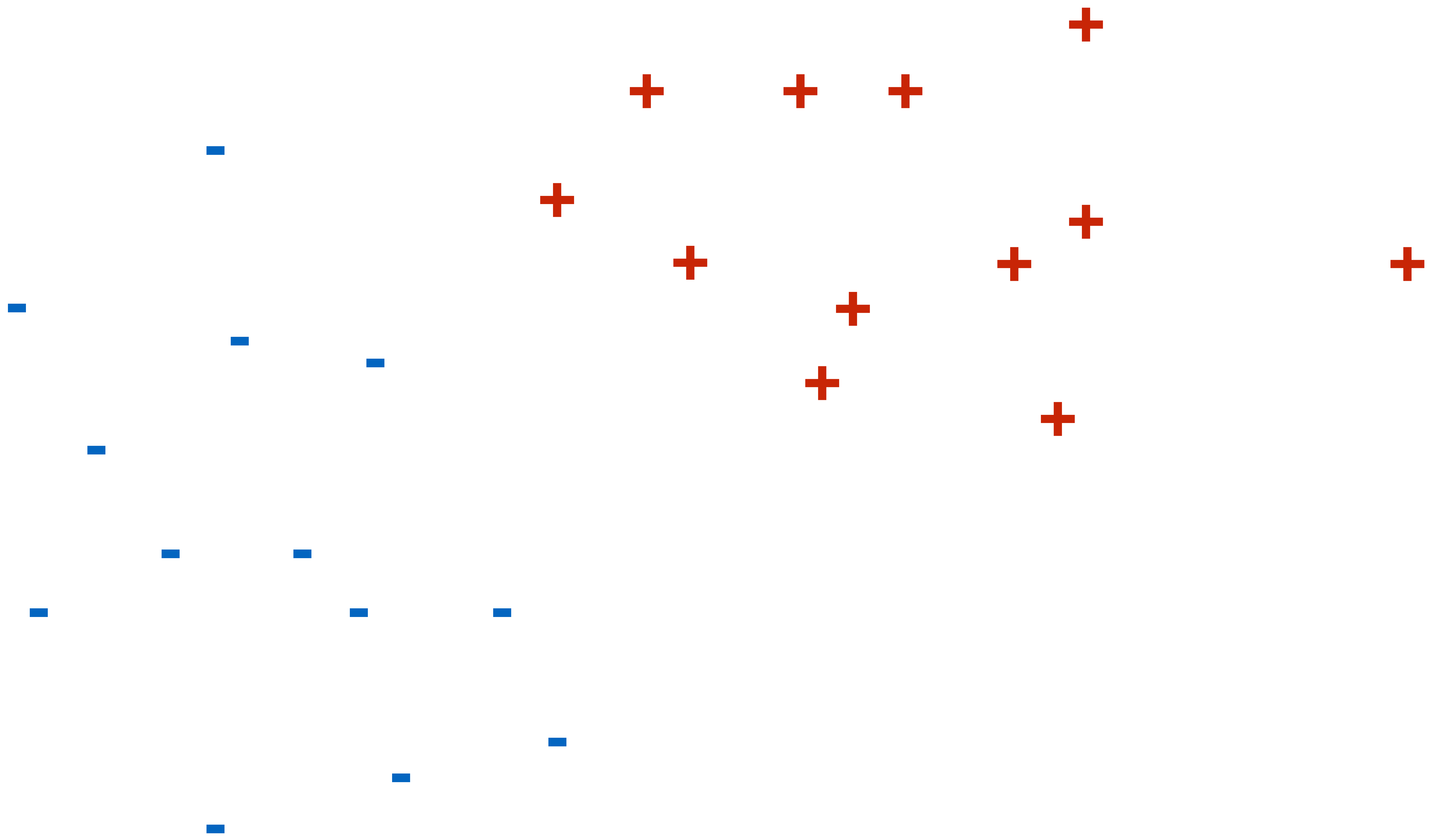

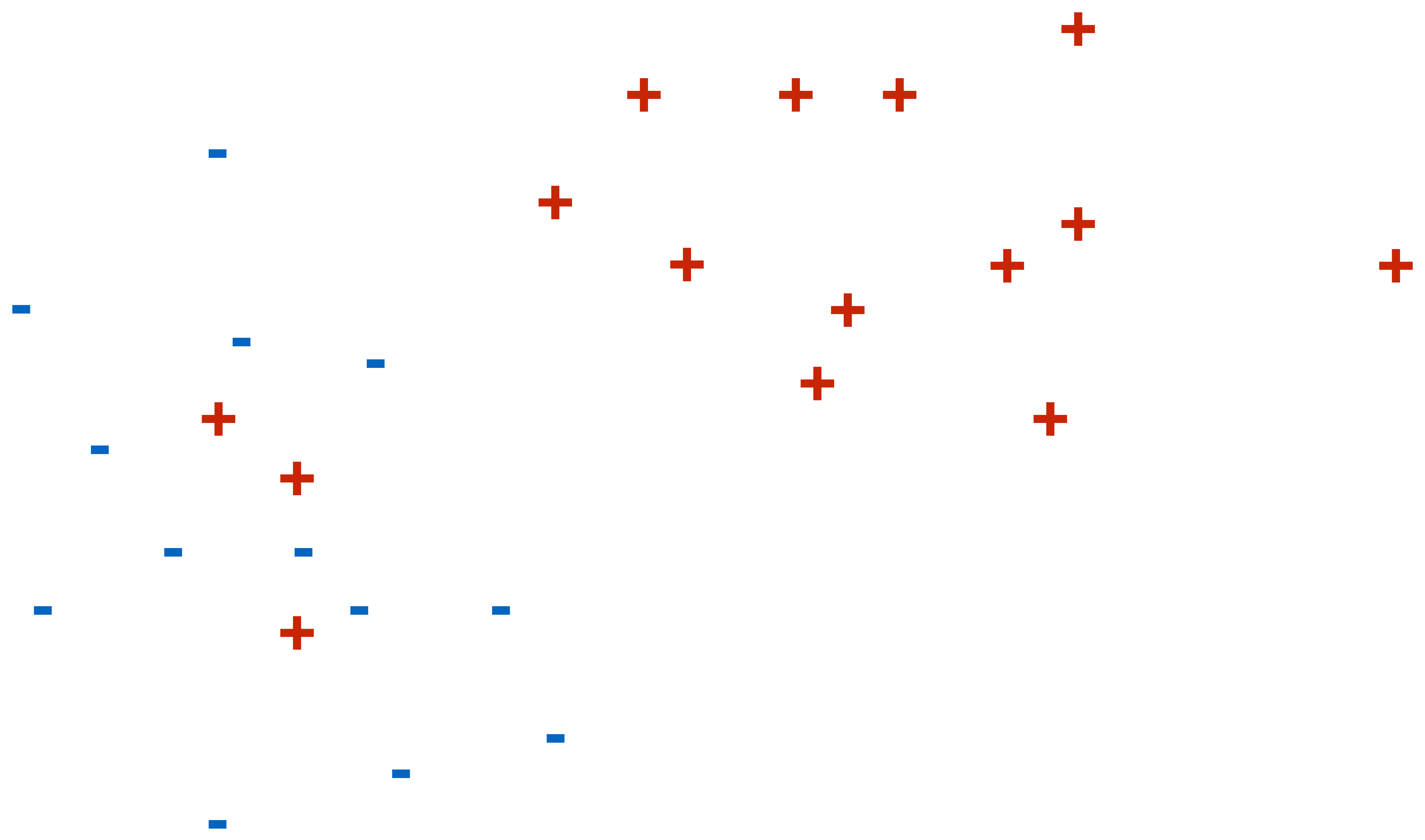

# Soft-Margin Form

$$\min_{w \in \mathbb{R}^d} \quad \frac{1}{2} w^\top w$$

$$\text{s.t.} \quad y_i(w^\top x_i + b) \geq 1 \qquad \forall i \in \{1, \ldots, n\}$$

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$$\min_{\substack{w \in \mathbb{R}^d \\ \xi \geq 0}} \quad \frac{1}{2} w^\top w$$

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$$\min_{\substack{w \in \mathbb{R}^d \\ \xi \geq 0}} \quad \frac{1}{2} w^\top w$$

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slack variables

# Soft-Margin Form

$$\min_{\substack{w \in \mathbb{R}^d \\ \xi \geq 0}} \quad \frac{1}{2} w^\top w + C \sum_{i=1}^{n} \xi_i$$

$$\text{s.t.} \quad y_i(w^\top x_i + b) \geq 1 - \xi_i \qquad \forall i \in \{1, \dots, n\}$$

slack variables

# Soft-Margin Form

slack penalty

$$\min_{\substack{w \in \mathbb{R}^d \\ \xi \geq 0}} \quad \frac{1}{2} w^\top w + C \sum_{i=1}^{n} \xi_i$$

$$\text{s.t.} \quad y_i(w^\top x_i + b) \geq 1 - \xi_i \qquad \forall i \in \{1, \ldots, n\}$$

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# Nonlinear Decision Boundary

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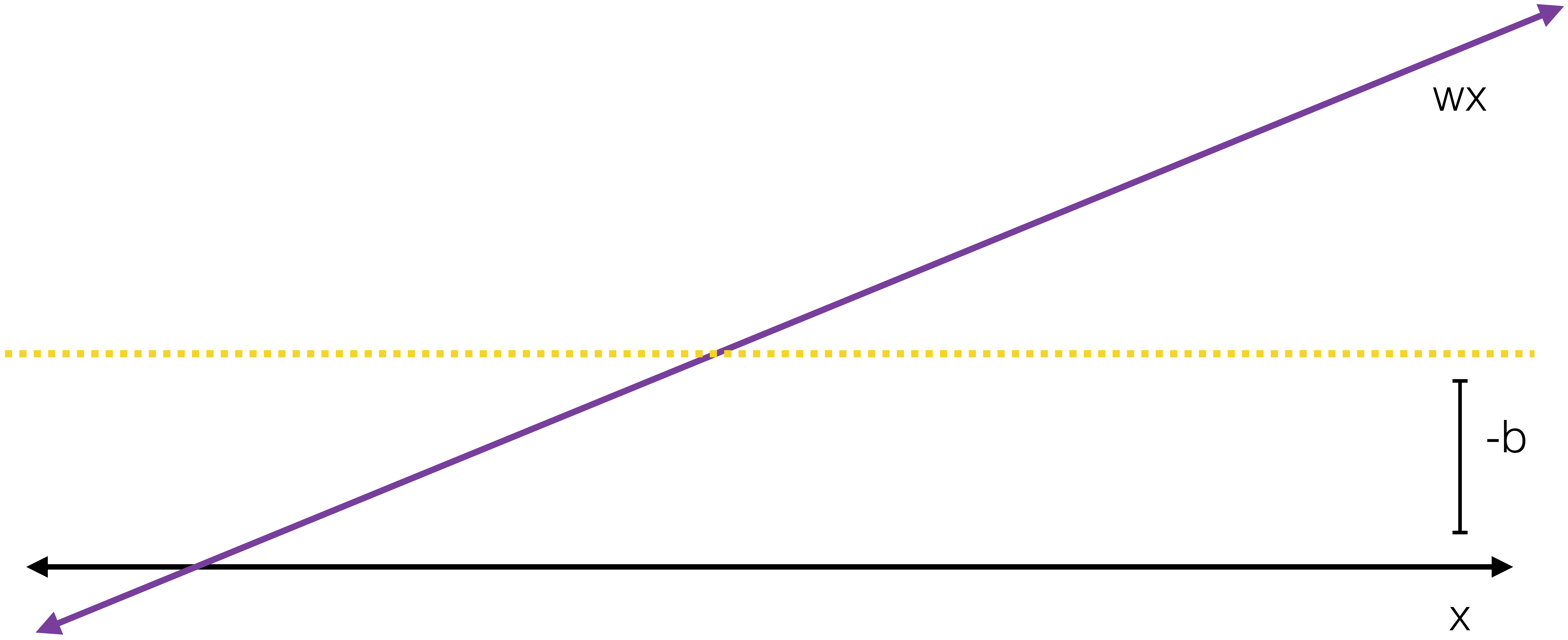

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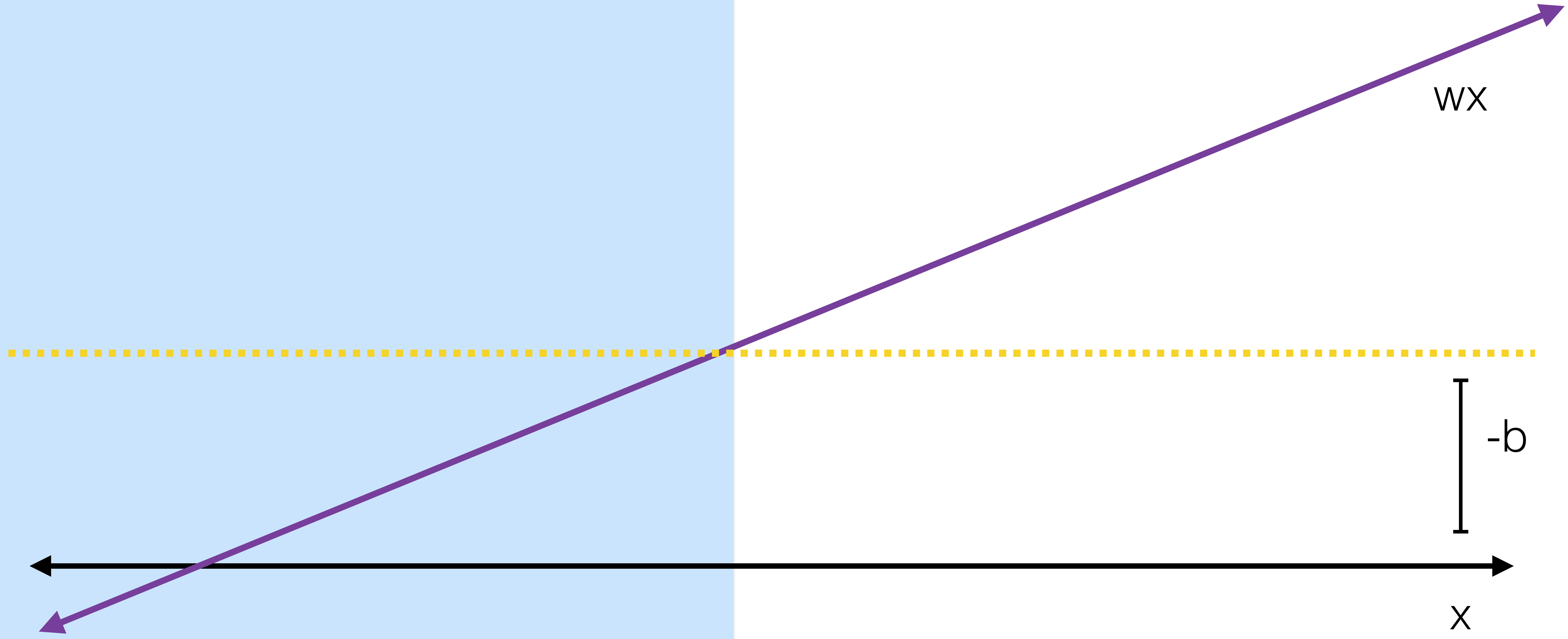

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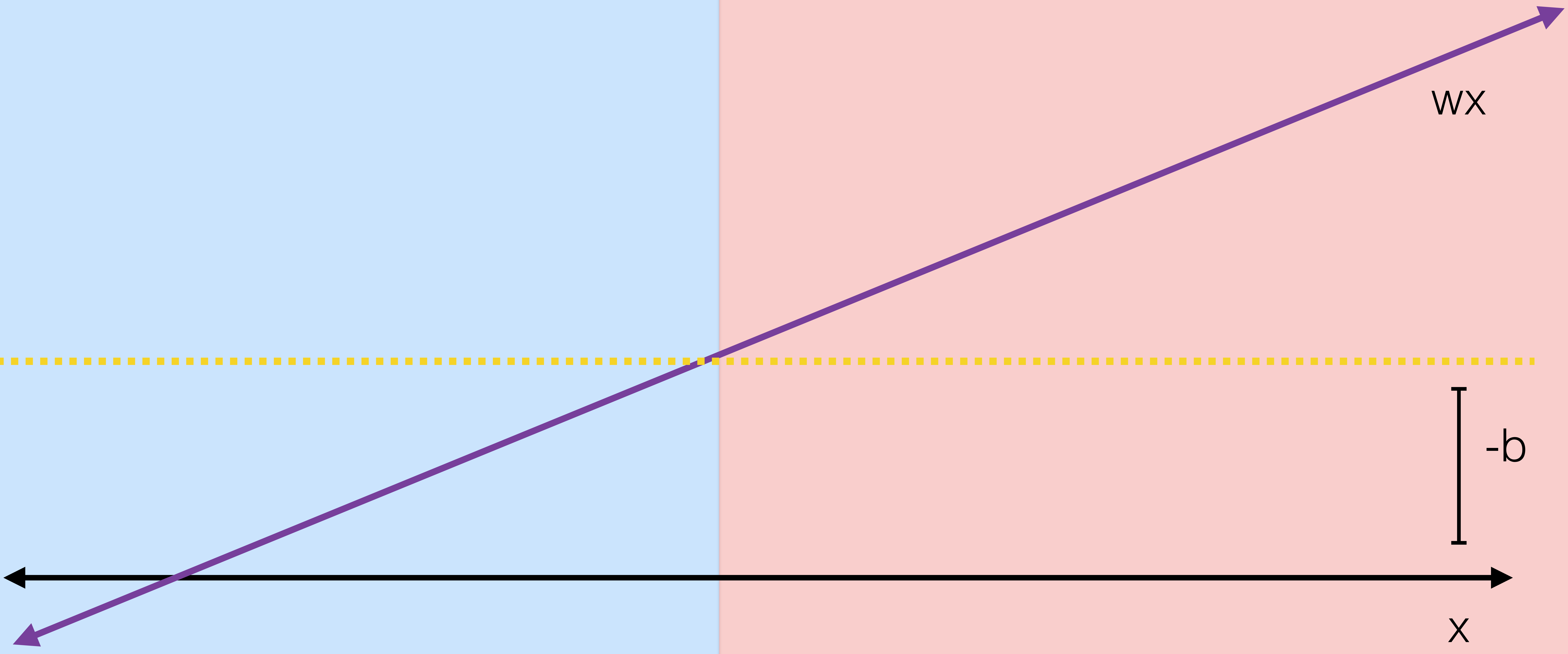
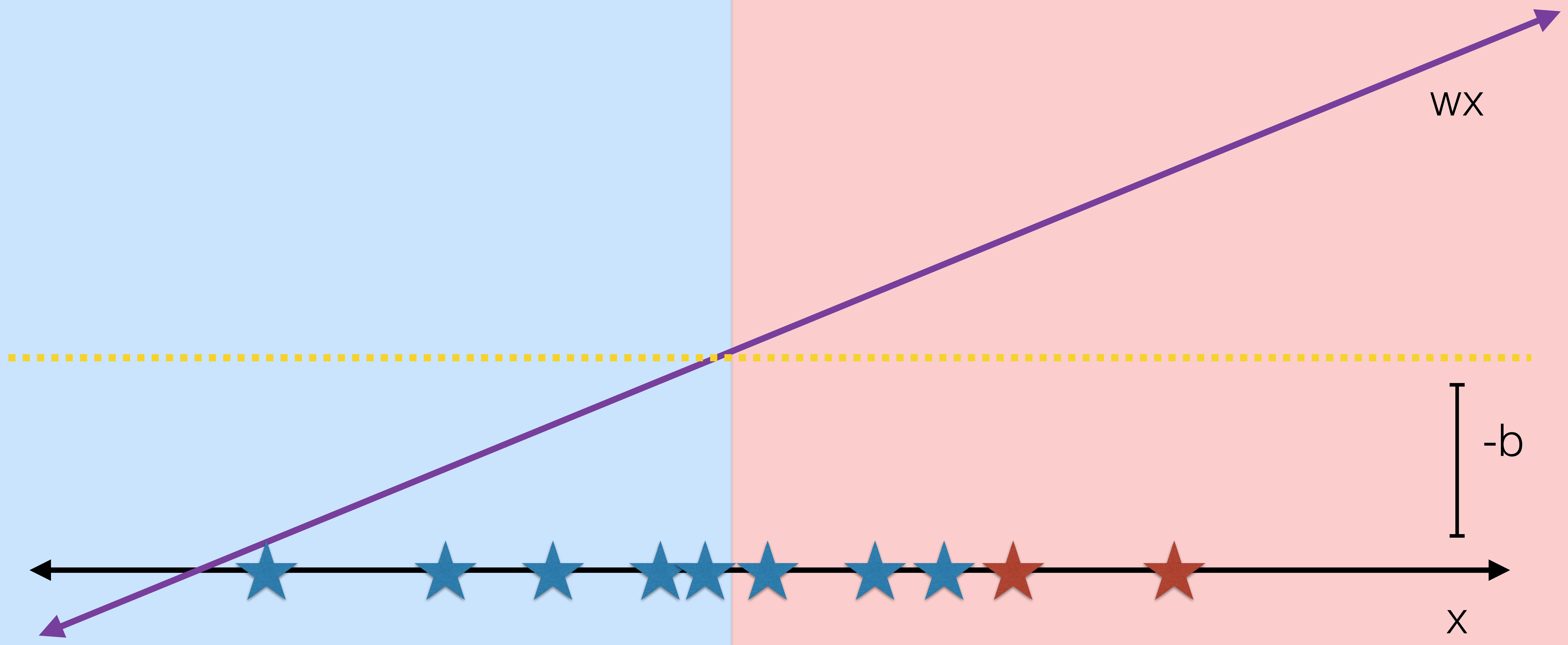


# Nonlinear Decision Boundary



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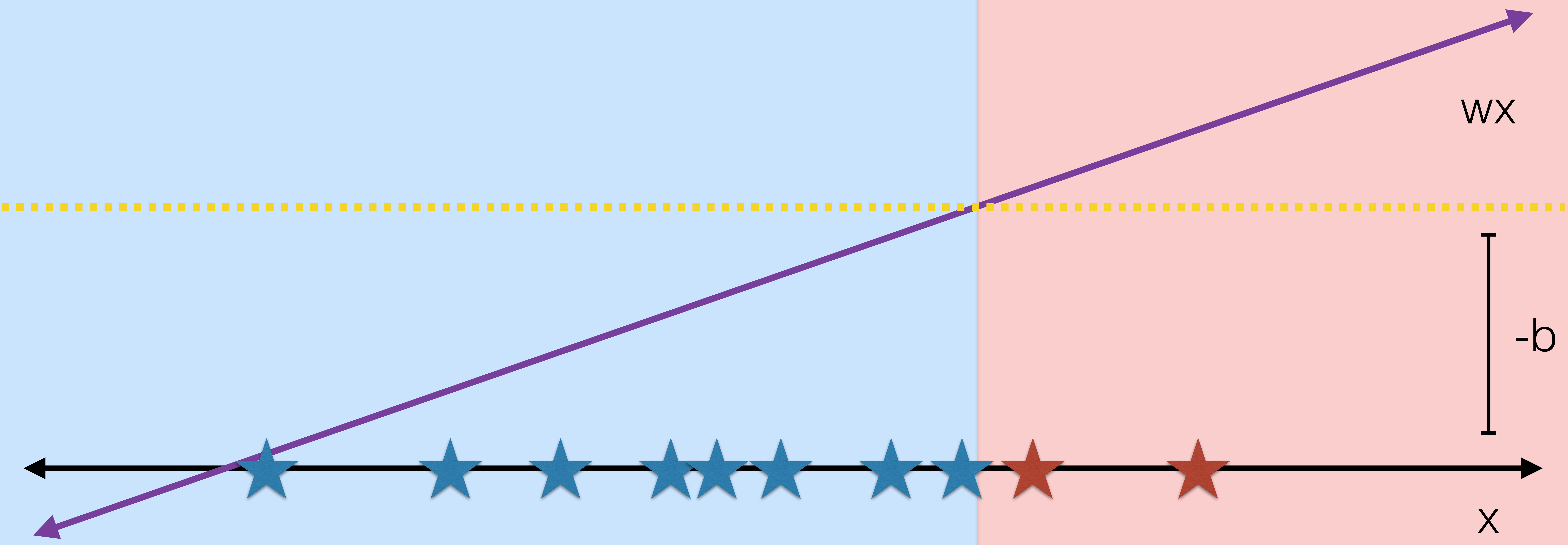

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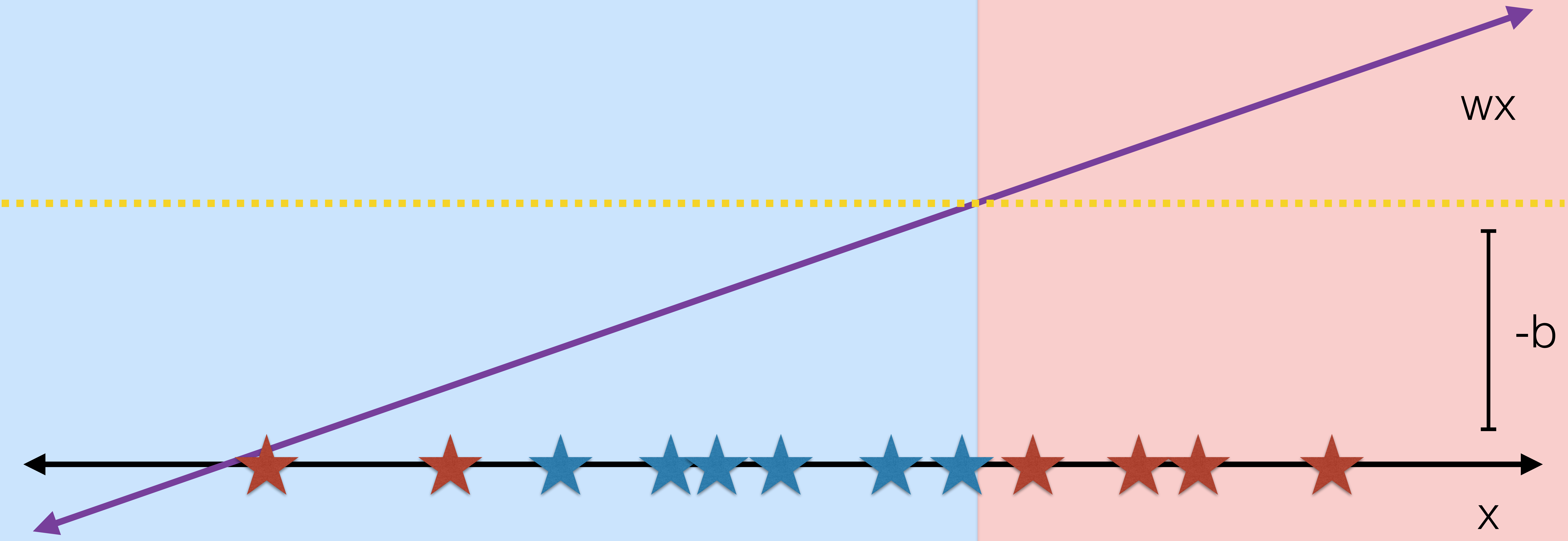

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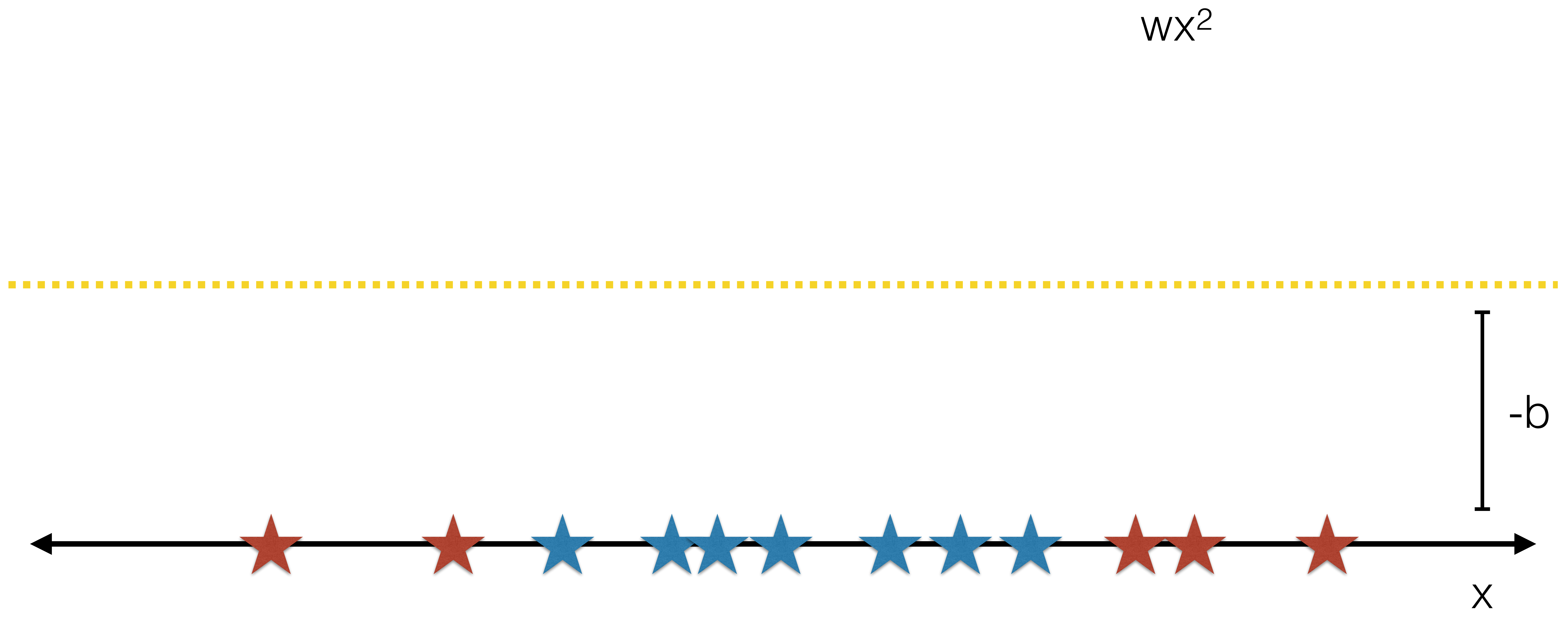

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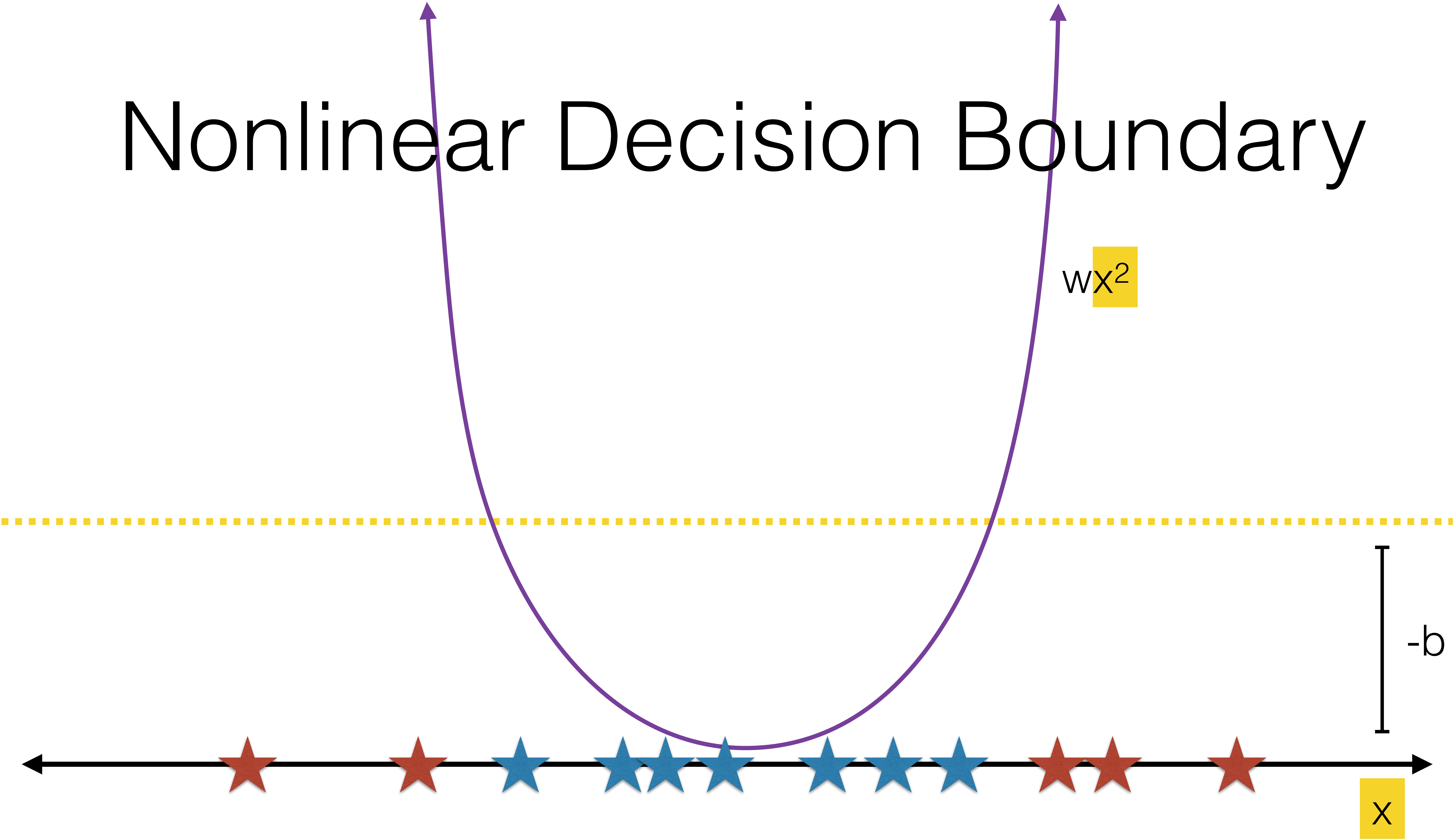

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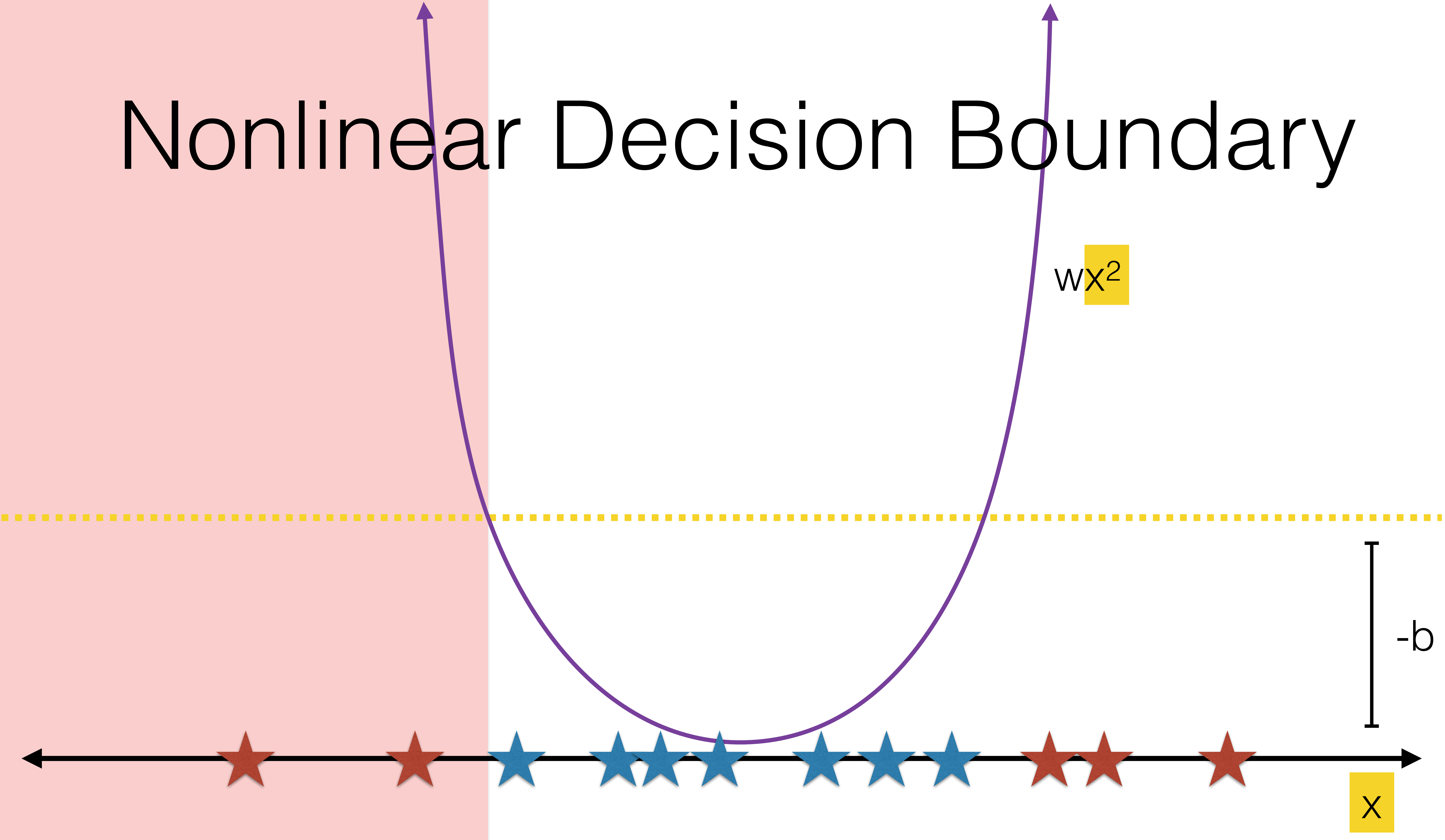

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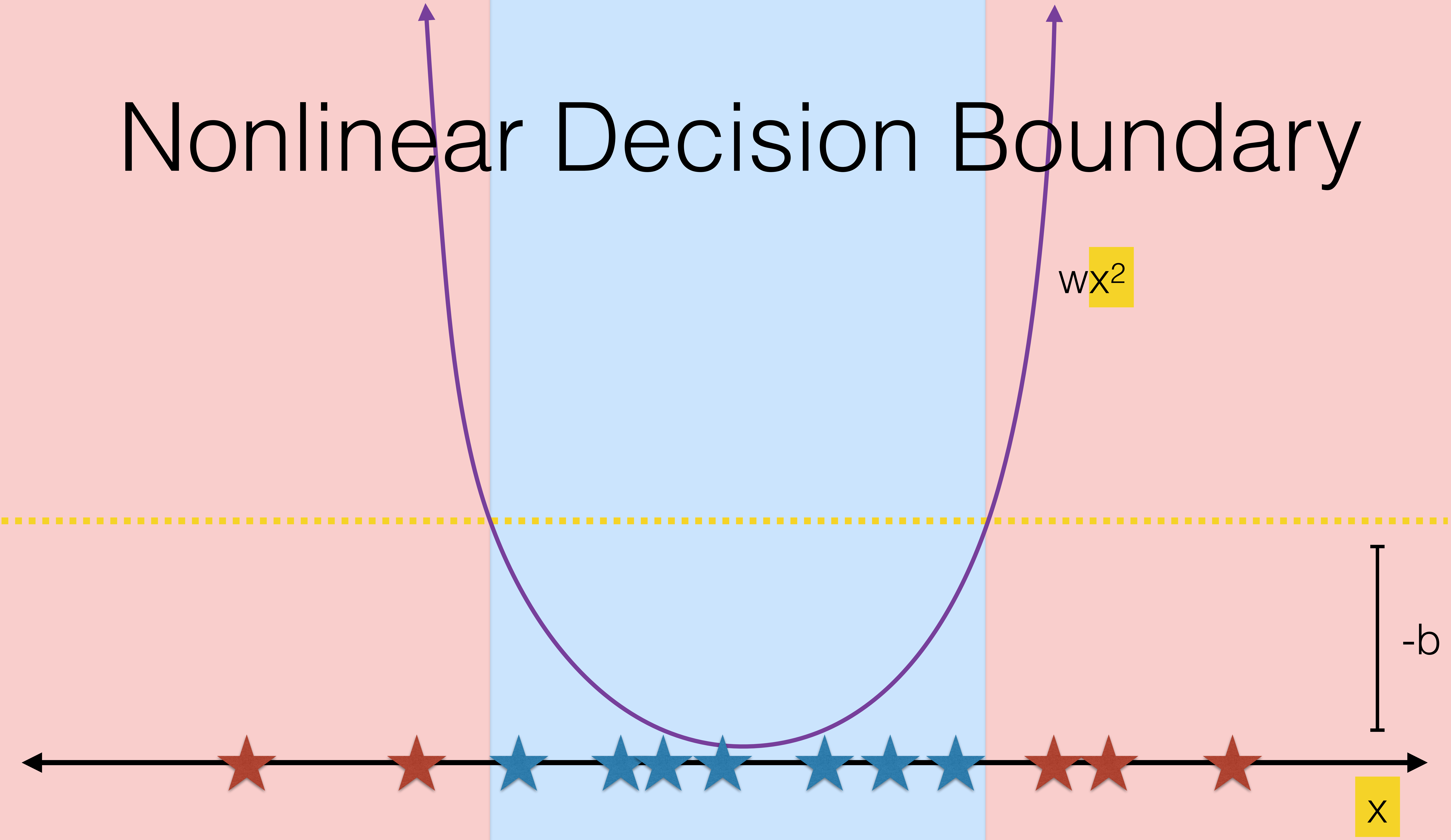

# Polynomial Feature Map

$$\Phi(x) = [x^1, \ldots, x^d, x^1 x^1, \ldots, x^1 x^d, \ldots, x^d x^1, \ldots, x^d x^d, \ldots]^\top$$

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Third-order terms

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Third-order terms $\{x^1x^1x^1, x^1x^1x^2, \ldots, x^1x^dx^d, \ldots, x^dx^dx^d\}$

Fourth-order terms $\{x^ix^jx^kx^\ell | i, j, k, \ell \in \{1, \ldots, d\}\}$

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$$|\Phi(x)| = \sum_{a=1}^{M} d^a = O(d^M)$$

# Soft-Margin Form w/ Feature Map

slack penalty

$$\min_{\substack{w \in \mathbb{R}^d \\ \xi \geq 0}} \quad \frac{1}{2} w^\top w + C \sum_{i=1}^{n} \xi_i$$

$$\text{s.t.} \quad y_i(w^\top \Phi(x_i) + b) \geq 1 - \xi_i \qquad \forall i \in \{1, \ldots, n\}$$

slack variables

(We rarely want to use this form.)

# Summary

- Large-margin and model complexity
- Formalizing large margin
- Quadratic program form
- Soft-margin
- Feature maps for non-linearity

# Optimization

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- “Off-the-shelf” quadratic programming solvers

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  - Leads to **kernel trick**