Greedy Graph Algorithms

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Graphs

- Model pairwise relationships (edges) between objects (nodes).
- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
- *Directed graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are ordered pairs.

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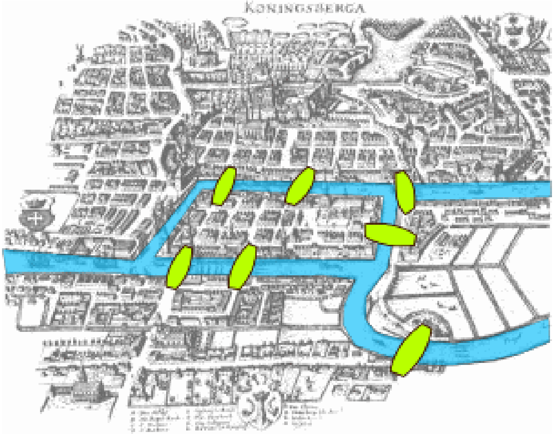

Shortest Path Problem

- $G(V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- *Length of a path* P is the sum of the lengths of the edges in P.
- Goal is to determine the shortest path from a specified start node s to each node in V.
- Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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SHORTEST PATHS

INSTANCE: A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION: A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Example of Dijkstra's Algorithm

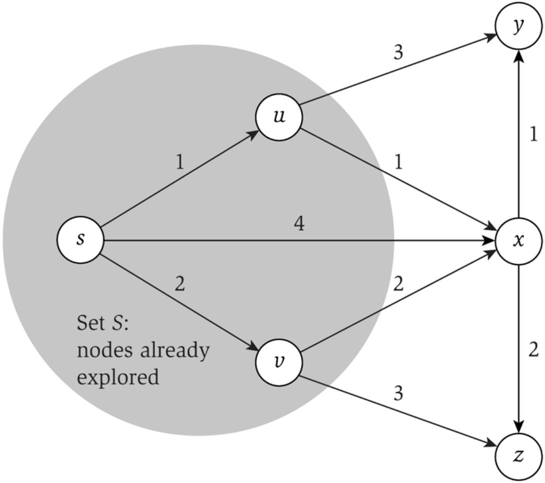

Figure 4.7 A snapshot of the execution of Dijkstra's Algorithm. The next node that will be added to the set S is x, due to the path through u.

Dijkstra's Algorithm

- Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u.
- "Greedily" add a node v to S that is closest to s.

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Dijkstra's Algorithm (G, ℓ)

Let S be the set of explored nodes

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Initially $S = \{s\}$ and $d(s) = 0$

While $S \neq V$

  Select a node $v \notin S$ with at least one edge from S for which

    $d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$ is as small as possible

  Add v to S and define $d(v) = d'(v)$

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- $d'(v) =$ length of shortest path from s to v using only nodes in S.
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- $d'(v)$ = length of shortest path from s to v using only nodes in S.
- To compute the shortest paths: store the predecessor u that minimises $d'(v)$.

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.

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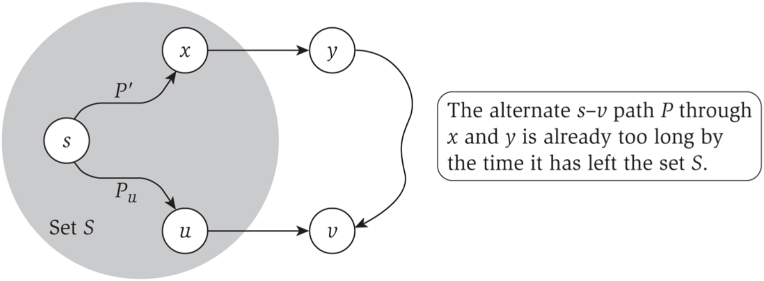

Figure 4.8 The shortest path P_v and an alternate s-v path P through the node y.

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- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
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Implementing Dijkstra's Algorithm

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Dijkstra's Algorithm (G, ℓ)
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While S ≠ V
    Select a node v ∉ S with at least one edge from S for which
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- Running time per iteration is $O(m)$, yielding an overall running time of $O(nm)$.

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- Determine the next node v to add to S using EXTRACTMIN.
- After adding v to S, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$,
  1. Set $d'(w) = d(v) + l_{(v,w)}$.
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- How many times are ExtractMin and ChangeKey invoked?

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively. Total running time is $O(m \log n)$.

Network Design

- Connect a set of nodes using a set of edges with certain properties.
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Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
- Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

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SOLUTION: A set $T \subseteq E$ of edges such that (V, T) is connected and the $\sum_{e \in T} c_e$ is as small as possible.

- Claim: If T is a minimum-cost solution to this network design problem then (V, T) is a tree.
- A subset T of E is a *spanning tree* of G if (V, T) is a tree.

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  Increasing cost order Process edges in increasing order of cost. Discard an edge if it creates a cycle.

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- Which of these algorithms works? All of them!
- Simplifying assumption: all edge costs are distinct.

Example of Prim's and Kruskal's Algorithms

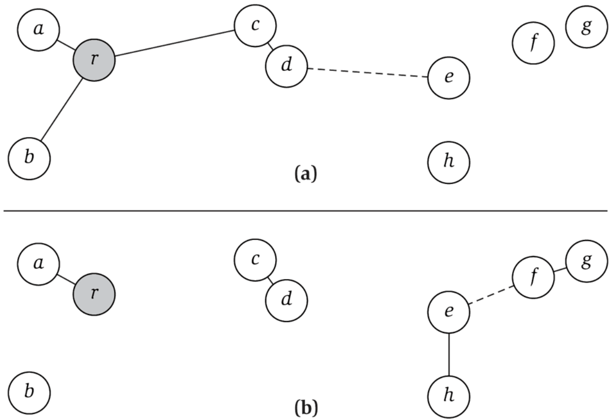

Figure 4.9 Sample run of the Minimum Spanning Tree Algorithms of (a) Prim and (b) Kruskal, on the same input. The first 4 edges added to the spanning tree are indicated by solid lines; the next edge to be added is a dashed line.

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  - Which edge should we add to join them?
- Which edges cannot belong to an MST?
  - What happens when we add an edge to an MST?
  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?

Graph Cuts

- A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
- Every set $S \subset V$ (S cannot be empty or the entire set V) has a corresponding cut: $\text{cut}(S)$ is the set of edges (v, w) such that $v \in S$ and $w \in V - S$.

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- $\text{cut}(S)$ is a cut because deleting the edges in $\text{cut}(S)$ disconnects S from $V - S$.

Cut Property

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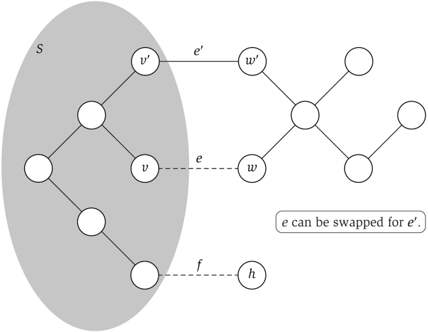

Figure 4.10 Swapping the edge e for the edge e' in the spanning tree T, as described in the proof of (4.17).

Optimality of Kruskal's Algorithm

- Kruskal's algorithm:
  - Start with an empty set T of edges.
  - Process edges in E in increasing order of cost.
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- Claim: Kruskal's algorithm outputs an MST.

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     - Why is e the cheapest edge in cut(S)?
  2. Prove that the algorithm computes a spanning tree.
     - (V, T) contains no cycles by construction.
     - If (V, T) is not connected, then exists a subset S of nodes not connected to $V - S$. What is the contradiction?

Optimality of Prim's Algorithm

- Prim's algorithm: Maintain a tree (S, U)
  - Start with an arbitrary node $s \in S$ and $U = \emptyset$.
  - Add the node v to S and the edge e to U that minimise

$$\min_{e=(u,v), u \in S, v \notin S} c_e \equiv \min_{e \in \text{cut}(S)} c_e.$$

  - Stop when $S = V$.
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     - In each iteration, S is the set added in the algorithm and e is the cheapest edge in cut(S) by construction.
  2. Prove that the graph constructed is a spanning tree.
     - Why are there no cycles in (V, T)?
     - Why is (V, T) connected?

Cycle Property

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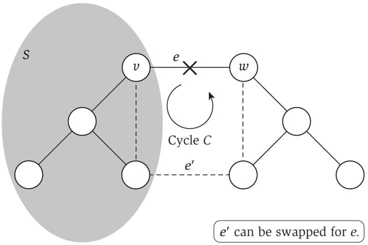

Figure 4.11 Swapping the edge e' for the edge e in the spanning tree T, as described in the proof of (4.20).

Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge e from E' only if (V, E') is connected after deletion.
  - Stop after processing all the edges.
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     - (V, E') is connected at the end, by construction.
     - If (V, E') contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.

Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!

Implementing Prim's Algorithm

- Maintain a tree (S, U).
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  - Stop when $S = V$.
- Sorting edges takes $O(m \log n)$ time.
- Implementation is very similar to Dijkstra's algorithm.
- Maintain S and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c_e$ for every node $v \in V - S$ in a priority queue.
- At each step, extract minimum v from priority queue and update the attachment costs of the neighbours of v.
- Total of $n - 1$ ExtractMin and m ChangeKey operations, yielding a running time of $O(m \log n)$.

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- Sorting edges takes $O(m \log n)$ time.
- Key question: "Does adding $e = (u, v)$ to T create a cycle?"
  - Maintain set of connected components of T.
  - $\textsc{Find}(u)$: return the name of the connected component of T that u belongs to.
  - $\textsc{Union}(A, B)$: merge connected components A and B.

Analysing Kruskal's Algorithm

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- Total running time of Kruskal's algorithm is $O(m \log n)$.

Comments on Union-Find and MST

- The UNION-FIND data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.

Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal's algorithm.
- Maintain disjoint subsets of elements from a universe U of n elements.
- Each subset has an name. We will set a set's name to be the identity of some element in it.
- Support three operations:
  1. MakeUnionFind(U): initialise the data structure with elements in U.
  2. Find(u): return the identity of the subset that contains u.
  3. Union(A, B): merge the sets named A and B into one set.

Union-Find Data Structure: Implementation 1

- Store all the elements of U in an array Component.
  - Assume identities of elements are integers from 1 to n.
  - Component[s] is the name of the set containing s.
- Implementing the operations:

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- Union is very slow because we cannot efficiently find the elements that belong to a set.

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array Elements
  - Indices of Elements range from 1 to n.
  - Elements$[s]$ stores the elements in the subset named s in a list.
- Execute Union(A, B) by merging B into A in two steps:
  1. Updating Component for elements of B in $O(|B|)$ time.
  2. Append Elements$[B]$ to Elements$[A]$ in $O(1)$ time.
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- Union takes $\Omega(n)$ in the worst-case.
- Optimisation 2: Store size of each set in an array (say, Size). If Size$[B] \leq$ Size$[A]$, merge B into A. Otherwise merge A into B. Update Size.

Union-Find Data Structure: Analysis of Implementation 2

- MakeUnionFind(S) and Find(u) are as before.

Union-Find Data Structure: Analysis of Implementation

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- Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?

Union-Find Data Structure: Implementation 3

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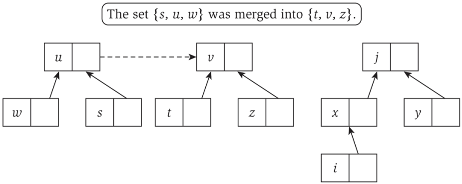

Figure 4.12 A `Union-Find` data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find`(i) would involve following the arrows i to x, and then x to j.

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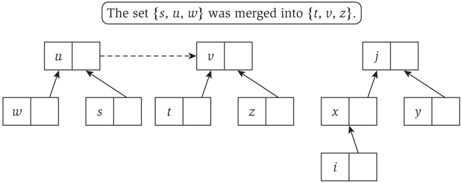

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- Implementing Find(u): follow pointers from u to the root of u's tree.
- Implementing Union(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.

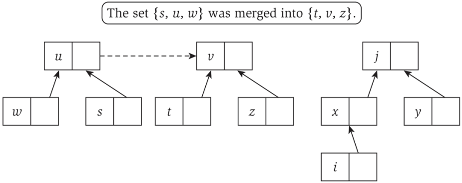


Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query Find(i) would involve following the arrows i to x, and then x to j.

Union-Find Data Structure: Find in Implementation 3

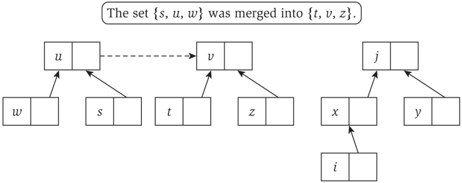
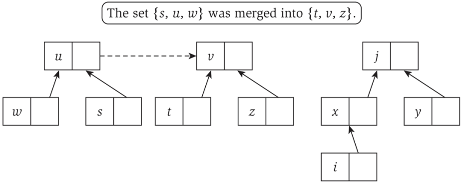


Figure 4.12 A `Union-Find` data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find`(i) would involve following the arrows i to x, and then x to j.

- Why does FIND(u) take $O(\log n)$ time?

Union-Find Data Structure: Find in Implementation 3



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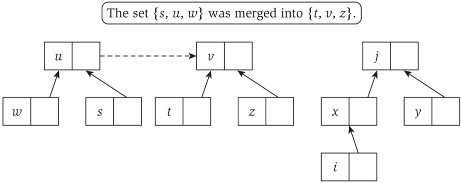
- Why does Find(u) take $O(\log n)$ time?
- Number of pointers followed equals the number of times the identity of the set containing u changed.
- Every time u's set's identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.

Union-Find Data Structure: Improving Implementation



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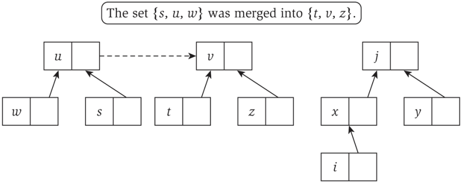
- Every time we invoke FIND(u), we follow the same set of pointers.

Union-Find Data Structure: Improving Implementation



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- Every time we invoke Find(u), we follow the same set of pointers.
- Path compression: make all nodes visited by Find(u) children of the root.

Union-Find Data Structure: Improving Implementation

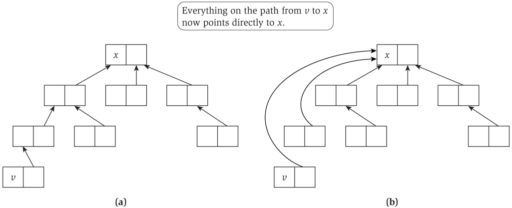

Figure 4.13 (a) An instance of a `Union-Find` data structure; and (b) the result of the operation `Find`(v) on this structure, using path compression.

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Union-Find Data Structure: Improving Implementation

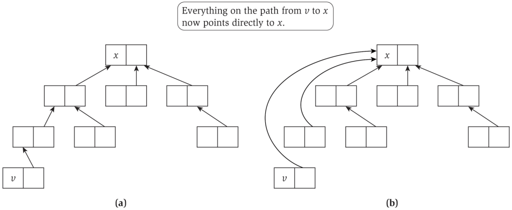

Figure 4.13 (a) An instance of a Union-Find data structure; and (b) the result of the operation Find(v) on this structure, using path compression.

- Every time we invoke FIND(u), we follow the same set of pointers.
- Path compression: make all nodes visited by FIND(u) children of the root.
- Can prove that total time taken by n FIND operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows e-x-t-r-e-m-e-l-y s-l-o-w-l-y with n.