NP-Complete Problems

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- If we use Karp reductions, we can refine the strategy:
  1. Prove that $X \in \mathcal{NP}$.
  2. Select a problem Y known to be $\mathcal{NP}$-Complete.
  3. Consider an arbitrary instance s_Y of problem Y. Show how to construct, in polynomial time, an instance s_X of problem X such that
     (a) If $s_Y \in Y$, then $s_X \in X$ and
     (b) If $s_X \in X$, then $s_Y \in Y$.

3-SAT is $\mathcal{NP}$-Complete

- Why is 3-SAT in NP?

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- Why is 3-SAT in NP?
- CIRCUIT SATISFIABILITY $\leq_P$ 3-SAT.
  1. Given an instance of CIRCUIT SATISFIABILITY, create an instance of SAT, in which each clause has *at most* three variables.
  2. Convert this instance of SAT into one of 3-SAT.

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- Output: if o is the output node, use the clause (x_o).

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  - If a clause has a single term t, replace the clause with $(t \vee z_1 \vee z_2)$.
  - If a clause has a two terms t and t', replace the clause with $t \vee t' \vee z_1$.

More $\mathcal{NP}$-Complete problems

- CIRCUIT SATISFIABILITY is $\mathcal{NP}$-Complete.
- We just showed that CIRCUIT SATISFIABILITY $\leq_P$ 3-SAT.
- We know that

3-SAT $\leq_P$ INDEPENDENT SET $\leq_P$ VERTEX COVER $\leq_P$ SET COVER

- All these problems are in $\mathcal{NP}$.
- Therefore, INDEPENDENT SET, VERTEX COVER, and SET COVER are $\mathcal{NP}$-Complete.

Hamiltonian Cycle

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- Another type of computationally hard problem involves searching over the set of all permutations of a collection of objects.
- In a directed graph $G(V, E)$, a cycle C is a *Hamiltonian cycle* if C visits each vertex exactly once.

HAMILTONIAN CYCLE

INSTANCE: A directed graph G.

QUESTION: Does G contain a Hamiltonian cycle?

Hamiltonian Cycle is $\mathcal{NP}$-Complete

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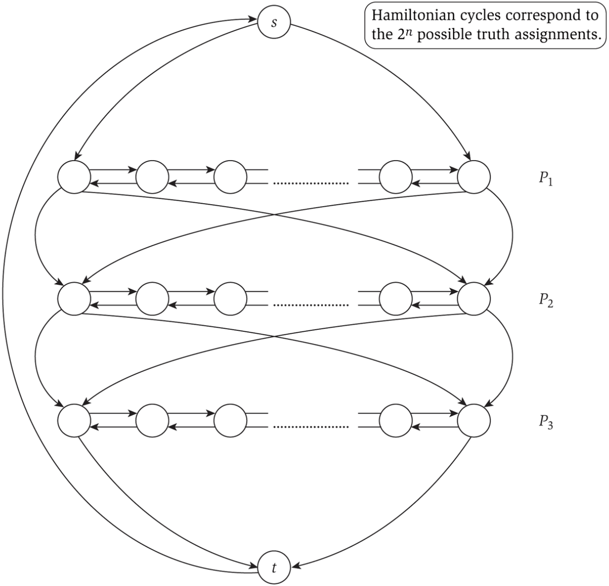
Hamiltonian Cycle is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Claim: $\text{3-SAT} \leq_P$ Hamiltonian Cycle.
- Consider an arbitrary instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots C_k$.
- Strategy:
  1. Construct a graph G with $O(nk)$ nodes and edges and 2^n Hamiltonian cycles with a one-to-one correspondence with 2^n truth assignments.
  2. Add nodes to impose constraints arising from clauses.
  3. Construction takes $O(nk)$ time.
- G contains n paths $P_1, P_2, \ldots P_n$.
- Each P_i contains $b = 3k + 3$ nodes $v_{i,1}, v_{i,2}, \ldots v_{i,b}$.

3-SAT $\leq_P$ Hamiltonian Cycle: Constructing G



3-SAT $\leq_P$ Hamiltonian Cycle: Modelling clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

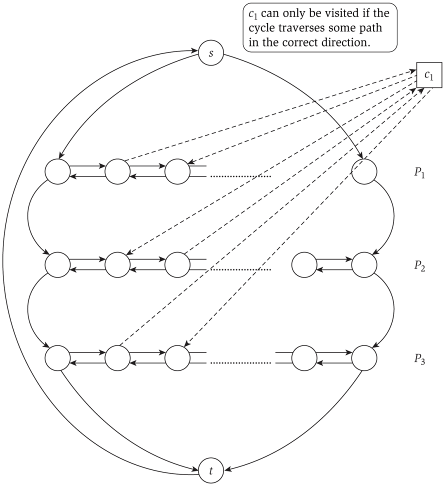


Figure 8.8 The reduction from 3-SAT to Hamiltonian Cycle: part 2.

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  - Otherwise, traverse P_i from right to left in $\mathcal{C}$.
  - For each clause C_j, there is at least one term set to 1. If the term is x_i, splice c_j into $\mathcal{C}$ using edge from $v_{i,3j}$ and edge to $v_{i,3j+1}$. Analogous construction if term is $\overline{x_i}$.

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- G has a Hamiltonian cycle $\mathcal{C} \rightarrow 3\text{-SAT}$ instance is satisfiable.
  - If $\mathcal{C}$ enters c_j on an edge from $v_{i,3j}$, it must leave c_j along the edge to $v_{i,3j+1}$.
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  - Nodes immediately before and after c_j in $\mathcal{C}$ are themselves connected by an edge e in G.
  - If we remove all such edges e from $\mathcal{C}$, we get a Hamiltonian cycle $\mathcal{C}'$ in $G - \{c_1, c_2, \ldots, c_k\}$.
  - Use $\mathcal{C}'$ to construct truth assignment to variables.
  - Argue that the assignment is a satisfying assignment.

The Traveling Salesman Problem

- A salesman must visit n cities $v_1, v_2, \ldots v_n$ starting at home city v_1.
- Salesman must find a *tour*, an order in which to visit each city exactly once, and return home.
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- For every pair of cities v_i and v_j, let $d(v_i, v_j) > 0$ be the distance from v_i to v_j.
- A *tour* is a permutation $v_{i_1} = v_1, v_{i_2}, \dots v_{i_n}$.
- The *length* of the tour is $\sum_{j=1}^{n-1} d(v_{i_j} v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$.

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TRAVELLING SALESMAN

INSTANCE: A set V of n cities, a function $d : V \times V \rightarrow \mathbb{R}^+$, and a number $D > 0$.

QUESTION: Is there a tour of length at most D?

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- Given a directed graph $G(V, E)$,
  - Create a city v_i for each node $i \in V$.
  - Define $d(v_i, v_j) = 1$ if $(i, j) \in E$.
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- Claim: G has a Hamiltonian cycle iff the instance of Travelling Salesman has a tour of length at most n.

Special Cases and Extensions that are $\mathcal{NP}$-Complete

- Hamiltonian Cycle for undirected graphs.
- Hamiltonian Path for directed and undirected graphs.
- Travelling Salesman with symmetric distances (by reducing Hamiltonian Cycle for undirected graphs to it).
- Travelling Salesman with distances defined by points on the plane.

3-Dimensional Matching

BIPARTITE MATCHING

INSTANCE: Disjoint sets X, Y, each of size n, and a set $T \subseteq X \times Y$ of pairs

QUESTION: Is there a set of n pairs in T such that each element of $X \cup Y$ is contained in exactly one of these pairs?

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- Easy to show 3-Dimensional Matching $\leq_P$ Set Cover and 3-Dimensional Matching $\leq_P$ Set Packing.

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Show that $3\text{-SAT} \leq_P 3\text{-DIMENSIONAL MATCHING}$.
- Strategy:
  - Start with an instance of 3-SAT with n variables and k clauses.
  - Create a gadget for each variable x_i that encodes the choice of truth assignment to x_i.
  - Add gadgets that encode constraints imposed by clauses.

3-SAT $\leq_P$ 3-Dimensional Matching: Variables

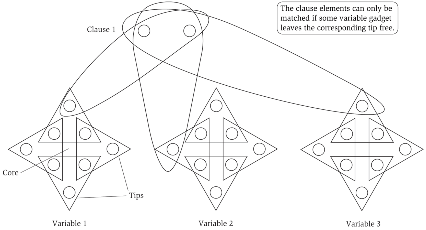

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Each x_i corresponds to a *variable gadget* i with $2k$ *core* elements $A_i = \{a_{i,1}, a_{i,2}, \ldots a_{i,2k}\}$ and $2k$ *tips* $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,2k}\}$.
- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.

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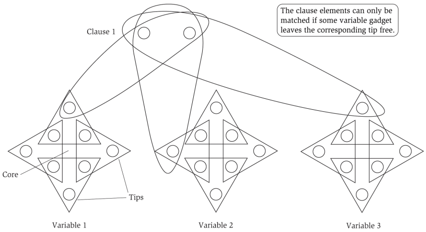

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- In any perfect matching, we can cover the elements in A_i

3-SAT $\leq_P$ 3-Dimensional Matching: Variables

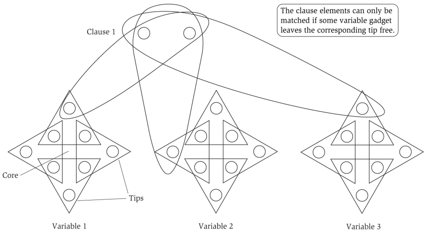


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Each x_i corresponds to a *variable gadget* i with $2k$ *core* elements $A_i = \{a_{i,1}, a_{i,2}, \ldots a_{i,2k}\}$ and $2k$ *tips* $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,2k}\}$.
- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.
- In any perfect matching, we can cover the elements in A_i either using all the even triples in gadget i or all the odd triples in the gadget.
- Even triples used, odd tips free $\equiv x_i = 0$; odd triples used, even tips free $\equiv x_i = 1$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

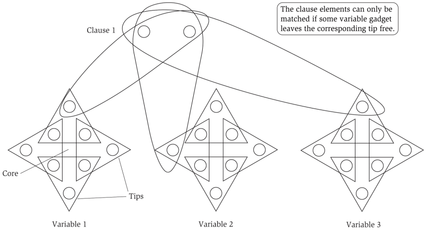


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

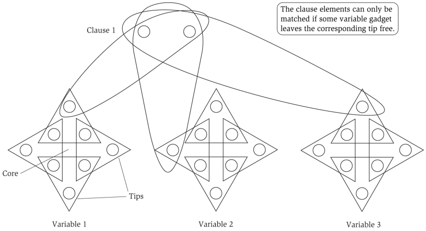
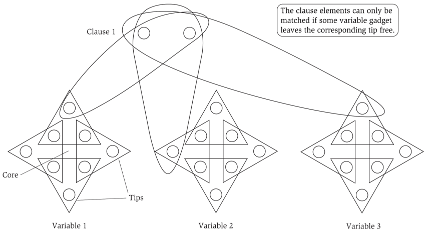


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses



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- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."
- *Clause gadget* j for clause C_j contains two core elements $P_j = \{p_j, p'_j\}$ and three triples:
  - C_j contains x_i: add triple $(p_j, p'_j, b_{i,2j})$.
  - C_j contains $\overline{x_i}$: add triple $(p_j, p'_j, b_{i,2j-1})$.

3-SAT $\leq_P$ 3-Dimensional Matching: Example

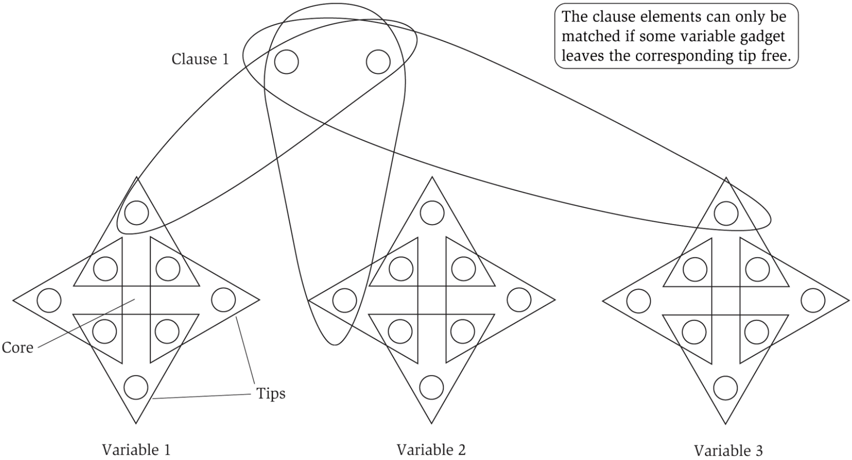

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.

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  - We have not covered all the tips!
  - Add $(n-1)k$ *cleanup gadgets* to allow the remaining $(n-1)k$ tips to be covered: cleanup gadget i contains two core elements $Q = \{q_i, q'_i\}$ and triple (q_i, q'_i, b) for every tip b in variable gadget i.

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- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} ($x_i = 0$) or all odd a_{ij} ($x_i = 1$).

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  - Is clause C_j satisfied?

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- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
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  - We have not covered all the tips!
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- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.
  - Is clause C_j satisfied? Core in clause gadget j is covered by some triple $\Rightarrow$ other element in the triple must be a tip element from the correct odd/even set in the three variable gadgets corresponding to a term in C_j.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

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- Did we create an instance of 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.
- How many elements do we have?
  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.

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  - k p_j elements.
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  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an instance of 3-Dimensional Matching?
- We need three sets X, Y, and Z of equal size.
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  - $2nk$ a_{ij} elements.
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  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
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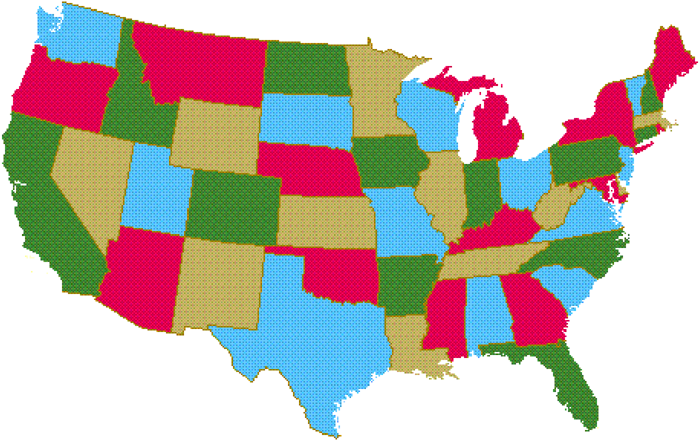
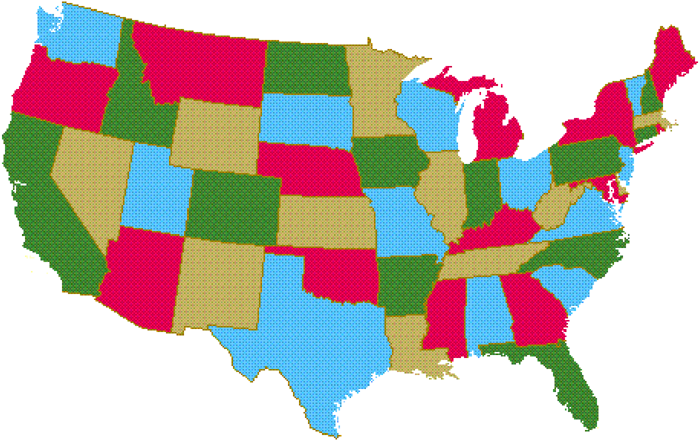
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.
- Each triple contains exactly one element from X, Y, and Z.

Colouring maps



Colouring maps



- Any map can be coloured with four colours (Appel and Hakken, 1976).

Graph Colouring

- Given an undirected graph $G(V, E)$, a *k-colouring* of G is a function $f : V \to \{1, 2, \ldots k\}$ such that for every edge $(u, v) \in E$, $f(u) \neq f(v)$.

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GRAPH COLOURING (k-COLOURING)

INSTANCE: An undirected graph $G(V, E)$ and an integer $k > 0$.

QUESTION: Does G have a k-colouring?

Applications of Graph Colouring

1. Job scheduling: assign jobs to n processors under constraints that certain pairs of jobs cannot be scheduled at the same time.
2. Compiler design: assign variables to k registers but two variables being used at the same time cannot be assigned to the same register.
3. Wavelength assignment: assign one of k transmitting wavelengths to each of n wireless devices. If two devices are close to each other, they must get different wavelengths.

2-Colouring

- How hard is 2-Colouring?

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- How hard is 2-Colouring?
- Claim: A graph is 2-colourable if and only if it is bipartite.
- Testing 2-colourability is possible in $O(|V| + |E|)$ time.
- What about 3-colouring? Is it easy to exhibit a certificate that a graph *cannot* be coloured with three colours?

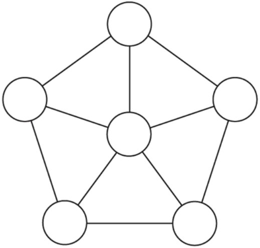

Figure 8.10 A graph that is not 3-colorable.

3-Colouring is $\mathcal{NP}$-Complete

- Why is 3-Colouring in $\mathcal{NP}$?

3-Colouring is $\mathcal{NP}$-Complete

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- 3-SAT $\leq_P$ 3-Colouring.

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

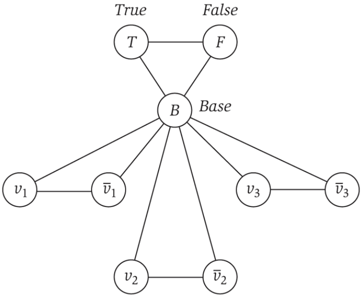


Figure 8.11 The beginning of the reduction for 3-Coloring.

- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

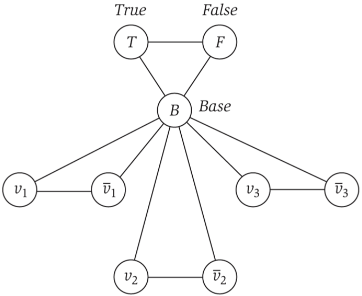


Figure 8.11 The beginning of the reduction for 3-Coloring.

- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.
- In any 3-Colouring, nodes v_i and $\overline{v_i}$ get a colour different from *Base*.
- *True colour*: colour assigned to the *True* node; *False colour*: colour assigned to the *False* node.
- Set x_i to 1 iff v_i gets the *True* colour.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

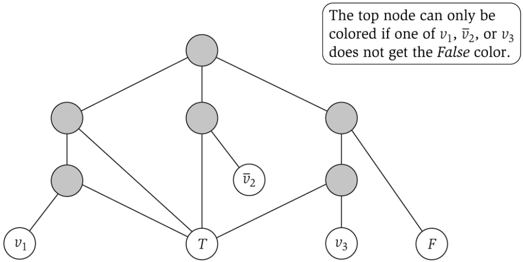
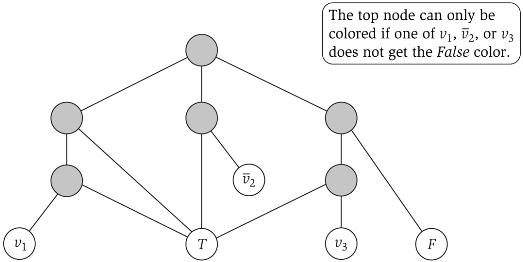


Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- Attach a six-node subgraph for this clause to the rest of the graph.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses



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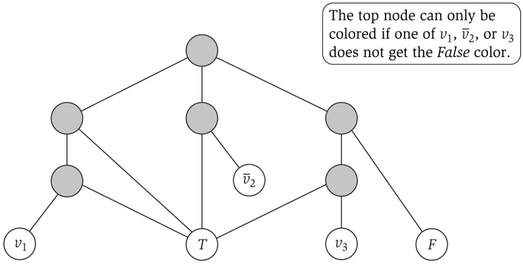
- Claim: Top node in the subgraph can be coloured in a 3-colouring iff one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses



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- Claim: Top node in the subgraph can be coloured in a 3-colouring iff one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour.
- Claim: Graph is 3-colourable iff instance of 3-SAT is satisfiable.

Subset Sum

SUBSET SUM

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$ and a target W.

QUESTION: Is there a subset of $\{w_1, w_2, \ldots, w_n\}$ whose sum is W?

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- Claim: SUBSET SUM is $\mathcal{NP}$-Complete, 3-DIMENSIONAL MATCHING $\leq_P$ SUBSET SUM.

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- Claim: SUBSET SUM is $\mathcal{NP}$-Complete,
  3-DIMENSIONAL MATCHING $\leq_P$ SUBSET SUM.
- Caveat: Special case of SUBSET SUM in which W is bounded by a polynomial function of n is not $\mathcal{NP}$-Complete (read pages 494–495 of your textbook).

Asymmetry of Certification

- Definition of efficient certification and $\mathcal{NP}$ is fundamentally asymmetric:
  - An input string s is a "yes" instance iff there exists a short string t such that $B(s, t) = \texttt{yes}$.
  - An input string s is a "no" instance iff *for all* short strings t, $B(s, t) = \texttt{no}$.

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  - An input string s is a "no" instance iff *for all* short strings t, $B(s, t) = \texttt{no}$. The definition of $\mathcal{NP}$ does not guarantee a short proof for "no" instances.

co-$\mathcal{NP}$

- For a decision problem X, its *complementary problem* $\overline{X}$ is the set of strings s such that $s \in \overline{X}$ iff $s \notin X$.

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- Open problem: Is $\mathcal{NP} = \text{co-}\mathcal{NP}$?
- Claim: If $\mathcal{NP} \neq \text{co-}\mathcal{NP}$ then $\mathcal{P} \neq \mathcal{NP}$.

Good Characterisations: the Class $\mathcal{NP} \cap$ co-$\mathcal{NP}$

- If a problem belongs to both $\mathcal{NP}$ and co-$\mathcal{NP}$, then
  - When the answer is yes, there is a short proof.
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- Claim: $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$.
- Open problem: Is $\mathcal{P} = \mathcal{NP} \cap$ co-$\mathcal{NP}$?