

Greedy Graph Algorithms

T. M. Murali

February 5, 10, and 12, 2009

Graphs

- ▶ Model pairwise relationships (edges) between objects (nodes).
- ▶ *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
- ▶ *Directed graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are ordered pairs.

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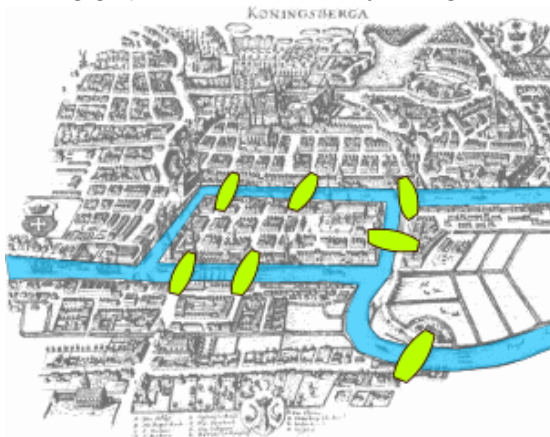
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Shortest Path Problem

- ▶ $G(V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- ▶ V has n nodes and E has m edges.
- ▶ *Length of a path P* is the sum of the lengths of the edges in P .
- ▶ Goal is to determine the shortest path from a specified start node s to each node in V .
- ▶ Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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SHORTEST PATHS

INSTANCE: A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION: A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u .

Example of Dijkstra's Algorithm

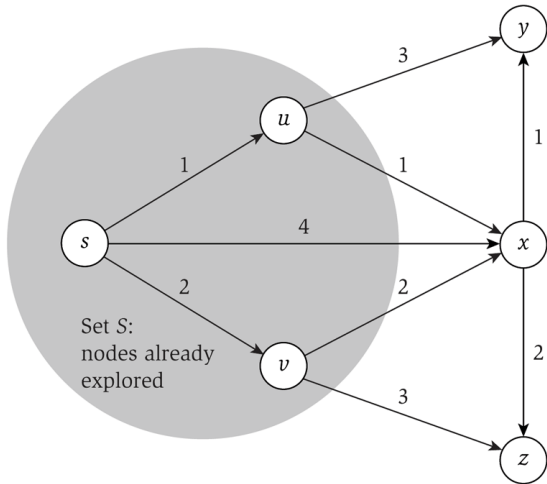


Figure 4.7 A snapshot of the execution of Dijkstra's Algorithm. The next node that will be added to the set S is x , due to the path through u .

Dijkstra's Algorithm

- ▶ Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u .
- ▶ “Greedy” add a node v to S that is closest to s .

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Dijkstra's Algorithm (G, ℓ)

Let S be the set of explored nodes

For each $u \in S$, we store a distance $d(u)$

Initially $S = \{s\}$ and $d(s) = 0$

While $S \neq V$

Select a node $v \notin S$ with at least one edge from S for which

$d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$ is as small as possible

Add v to S and define $d(v) = d'(v)$

EndWhile

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EndWhile

- ▶ Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimises $d'(v)$.

Proof of Correctness

- ▶ Let P_u be the shortest path computed for a node u .
- ▶ Claim: P_u is the shortest path from s to u .
- ▶ Prove by induction on the size of S .

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 - ▶ Inductive step: we add the node v to S . Let u be the v 's predecessor on the path P_v . Could there be a shorter path P from s to v ?

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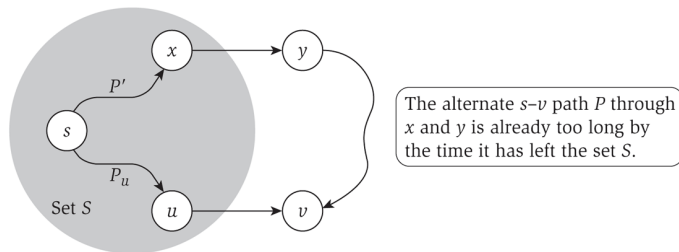


Figure 4.8 The shortest path P_v and an alternate s - v path P through the node y .

Comments about Dijkstra's Algorithm

- ▶ Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- ▶ Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

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- ▶ In each iteration, for each node $v \notin S$, compute

$$\min_{e=(u,v), u \in S} d(u) + \ell_e$$

- ▶ Running time per iteration is $O(m)$, yielding an overall running time of $O(nm)$.

A Faster implementation of Dijkstra's Algorithm

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- ▶ Observation: If we add v to S , $d'(w)$ changes only for v 's neighbours.
- ▶ Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- ▶ Determine the next node v to add to S using EXTRACTMIN.
- ▶ After adding v to S , for each neighbour w of v , compute $d(v) + l_{(v,w)}$.
- ▶ If $d(v) + l_{(v,w)} < d'(w)$,
 1. Set $d'(w) = d(v) + l_{(v,w)}$.
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- ▶ How many times are EXTRACTMIN and CHANGEKEY invoked? $n - 1$ and m times, respectively. Total running time is $O(m \log n)$.

Network Design

- ▶ Connect a set of nodes using a set of edges with certain properties.
- ▶ Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- ▶ Example: connect all nodes using a cycle of shortest total length.

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Minimum Spanning Tree (MST)

- ▶ Given an undirected graph $G(V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
- ▶ Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

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MINIMUM SPANNING TREE

INSTANCE: An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

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SOLUTION: A set $T \subseteq E$ of edges such that (V, T) is connected and the $\sum_{e \in T} c_e$ is as small as possible.

- ▶ Claim: If T is a minimum-cost solution to this network design problem then (V, T) is a tree.
- ▶ A subset T of E is a *spanning tree* of G if (V, T) is a tree.

Greedy Algorithm for the MST Problem

- ▶ Template: process edges in some order. Add an edge to T if tree property is not violated.

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Increasing cost order Process edges in increasing order of cost. Discard an edge if it creates a cycle.

Dijkstra-like Start from a node s and grow T outward from s : add the node that can be attached most cheaply to current tree.

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- ▶ Which of these algorithms works? All of them!
- ▶ Simplifying assumption: all edge costs are distinct.

Example of Prim's and Kruskal's Algorithms

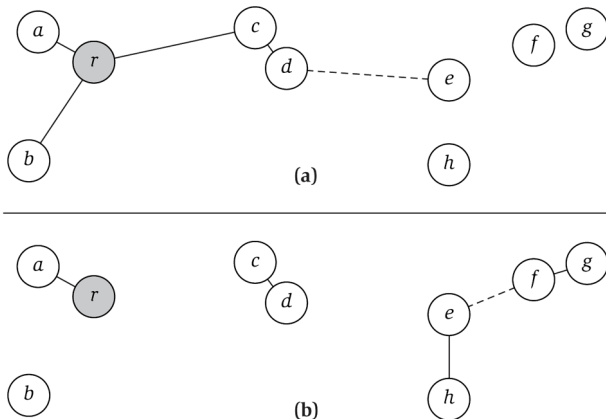


Figure 4.9 Sample run of the Minimum Spanning Tree Algorithms of (a) Prim and (b) Kruskal, on the same input. The first 4 edges added to the spanning tree are indicated by solid lines; the next edge to be added is a dashed line.

Graph Cuts

- ▶ A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
- ▶ Every set $S \subset V$ (S cannot be empty or the entire set V) has a corresponding cut: $\text{cut}(S)$ is the set of edges (v, w) such that $v \in S$ and $w \in V - S$.

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- ▶ $\text{cut}(S)$ is a cut because deleting the edges in $\text{cut}(S)$ disconnects S from $V - S$.

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- ▶ Proof: exchange argument. If a supposed MST T does not contain e , show that there is a tree with smaller cost than T that contains e .

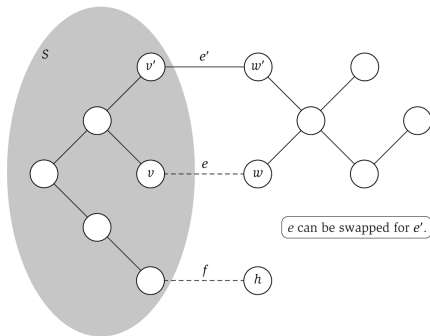


Figure 4.10 Swapping the edge e for the edge e' in the spanning tree T , as described in the proof of (4.17).

Using the Cut Property

- ▶ Let F be the set of all edges that satisfy the cut property.
- ▶ Is the graph induced by F connected?
- ▶ Can the graph induced by F contain a cycle?
- ▶ How many edges can F contain?

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- ▶ How many edges can F contain? $n - 1$
- ▶ F is the unique MST (problem 8, Chapter 4 of textbook).
- ▶ Kruskal's and Prim's algorithms compute F efficiently.

Optimality of Kruskal's Algorithm

- ▶ Kruskal's algorithm:
 - ▶ Start with an empty set T of edges.
 - ▶ Process edges in E in increasing order of cost.
 - ▶ Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.
- ▶ Claim: Kruskal's algorithm outputs an MST.

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 1. For every edge e added, demonstrate the existence of S and $V - S$ such that e and S satisfy the cut property.
 2. Prove that the algorithm computes a spanning tree.

Optimality of Prim's Algorithm

- ▶ Prim's algorithm: Maintain a tree (S, U)
 - ▶ Start with an arbitrary node $s \in S$ and $U = \emptyset$.
 - ▶ Add the node v to S and the edge e to U that minimise

$$\min_{e=(u,v), u \in S, v \notin S} c_e \equiv \min_{e \in \text{cut}(S)} c_e.$$

- ▶ Stop when $S = V$.
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- ▶ Stop when $S = V$.
- ▶ Claim: Prim's algorithm outputs an MST.
 1. Prove that every edge inserted satisfies the cut property.
 2. Prove that the graph constructed is a spanning tree.

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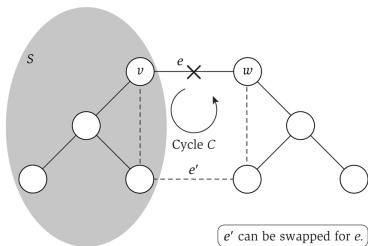


Figure 4.11 Swapping the edge e' for the edge e in the spanning tree T , as described in the proof of (4.20).

Optimality of the Reverse-Delete Algorithm

- ▶ Reverse-Delete algorithm: Maintain a set E' of edges.
 - ▶ Start with $E' = E$.
 - ▶ Process edges in decreasing order of cost.
 - ▶ Delete the next edge e from E' only if (V, E') is connected after deletion.
 - ▶ Stop after processing all the edges.
- ▶ Claim: the Reverse-Delete algorithm outputs an MST.

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 - ▶ Stop after processing all the edges.
- ▶ Claim: the Reverse-Delete algorithm outputs an MST.
 1. Show that every edge deleted belongs to no MST.
 2. Prove that the graph remaining at the end is a spanning tree.

Comments on MST Algorithms

- ▶ To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- ▶ *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!

Implementing Prim's Algorithm

- ▶ Maintain a tree (S, U) .
 - ▶ Start with an arbitrary node $s \in V$ and $U = \emptyset$.
 - ▶ Add the node v to S and the edge e to U that minimise

$$\min_{e \in \text{cut}(S)} c_e.$$

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- ▶ Stop when $S = V$.
- ▶ Sorting edges takes $O(m \log n)$ time.
- ▶ Implementation is very similar to Dijkstra's algorithm.
- ▶ Maintain S and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c_e$ for every node $v \in V - S$ in a priority queue.
- ▶ At each step, extract minimum v from priority queue and update the attachment costs of the neighbours of v .
- ▶ Total of $n - 1$ EXTRACTMIN and m CHANGEKEY operations, yielding a running time of $O(m \log n)$.

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- ▶ Sorting edges takes $O(m \log n)$ time.
- ▶ Key question: "Does adding $e = (u, v)$ to T create a cycle?"
 - ▶ Maintain set of connected components of T .
 - ▶ $\text{FIND}(u)$: return the name of the connected component of T that u belongs to.
 - ▶ $\text{UNION}(A, B)$: merge connected components A and B .
- ▶ Answering the question: Adding e creates a cycle if and only if $\text{FIND}(u) = \text{FIND}(v)$. If not, execute $\text{UNION}(\text{FIND}(u), \text{FIND}(v))$.

Analysing Kruskal's Algorithm

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 - ▶ Each FIND takes $O(1)$ time, k invocations of UNION take $O(k \log k)$ time in total.
 - ▶ Each FIND takes $O(\log n)$ time and each invocation of UNION takes $O(1)$ time.

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- ▶ Total running time of Kruskal's algorithm is $O(m \log n)$.

Union-Find Data Structure

- ▶ Abstraction of the data structure needed by Kruskal's algorithm.
- ▶ Maintain disjoint subsets of elements from a universe U of n elements.
- ▶ Each subset has a name. We will set a set's name to be the identity of some element in it.
- ▶ Support three operations:
 1. $\text{MAKEUNIONFIND}(U)$: initialise the data structure with elements in U .
 2. $\text{FIND}(u)$: return the identity of the subset that contains u .
 3. $\text{UNION}(A, B)$: merge the sets named A and B into one set.

Union-Find Data Structure: Implementation 1

- ▶ Store all the elements of U in an array `COMPONENT`.
 - ▶ Assume identities of elements are integers from 1 to n .
 - ▶ `COMPONENT[s]` is the name of the set containing s .
- ▶ Implementing the operations:

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 1. `MAKEUNIONFIND(U)`: For each $s \in U$, set `COMPONENT[s] = s` in $O(n)$ time.
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- ▶ `UNION` is very slow because we cannot efficiently find the elements that belong to a set.

Union-Find Data Structure: Implementation 2

- ▶ Optimisation 1: Use an array `ELEMENTS`
 - ▶ Indices of `ELEMENTS` range from 1 to n .
 - ▶ `ELEMENTS[s]` stores the elements in the subset named s in a list.
- ▶ Execute `UNION(A, B)` by merging B into A in two steps:
 1. Updating `COMPONENT` for elements of B in $O(|B|)$ time.
 2. Append `ELEMENTS[B]` to `ELEMENTS[A]` in $O(1)$ time.
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- ▶ Optimisation 2: Store size of each set in an array (say, `SIZE`). If $SIZE[B] \leq SIZE[A]$, merge B into A . Otherwise merge A into B . Update `SIZE`.

Union-Find Data Structure: Analysis of Implementation 2

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- ▶ FIND is fast in the worst case, UNION is fast in an amortised sense. Can we make both operations worst-case efficient?

Union-Find Data Structure: Implementation 3

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- ▶ Represent each subset in a tree using pointers:
 - ▶ Each tree node contains an element and a pointer to a parent.
 - ▶ The identity of the set is the identity of the element at the root.

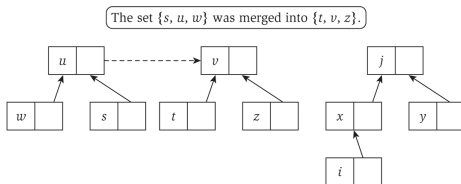


Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j . The dashed arrow from u to v is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query $\text{Find}(i)$ would involve following the arrows i to x , and then x to j .

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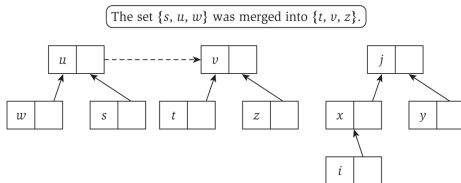


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- ▶ Implementing FIND(u): follow pointers from u to the root of u 's tree.
- ▶ Implementing UNION(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.

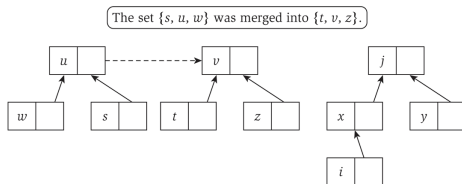


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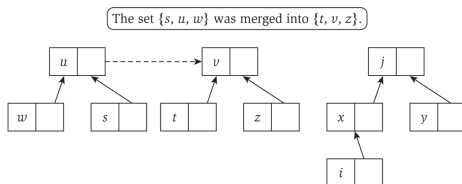


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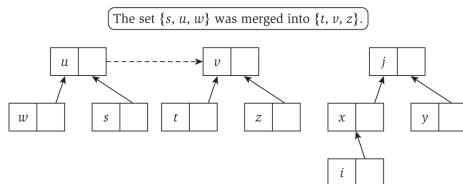


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- ▶ Why does $\text{FIND}(u)$ take $O(\log n)$ time?
- ▶ Number of pointers followed equals the number of times the identity of the set containing u changed.
- ▶ Every time u 's set's identity changes, the set at least doubles in size \Rightarrow there are $O(\log n)$ pointers followed.

Union-Find Data Structure: Improving Implementation 3

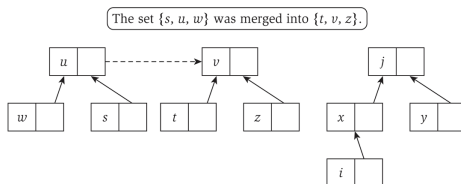


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- ▶ Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.

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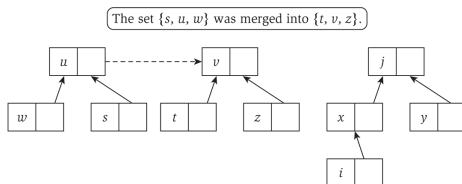


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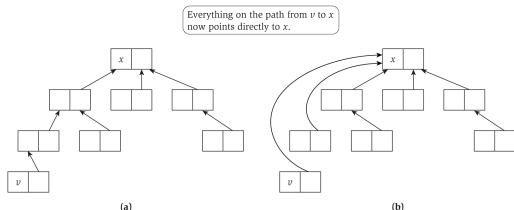


Figure 4.13 (a) An instance of a Union-Find data structure; and (b) the result of the operation $\text{Find}(v)$ on this structure, using path compression.

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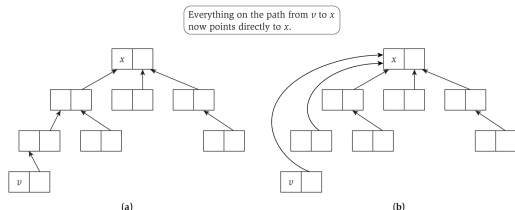


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- ▶ Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.
- ▶ Path compression: make all nodes visited by $\text{FIND}(u)$ children of the root.
- ▶ Can prove that total time taken by n FIND operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows e-x-t-r-e-m-e-l-y s-l-o-w-l-y with n .

Comments on Union-Find and MST

- ▶ The UNION-FIND data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- ▶ The data structure does not support edge **deletion** efficiently.
- ▶ Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- ▶ Holy grail: $O(m)$ deterministic algorithm for MST.