$2^k$ Factorial Designs

A $2^k$ experimental design is used to determine the effect of $k$ factors, each of which have two alternatives or levels.

- Used to prune out the less important factors for further study
- Pick the highest and lowest levels for each factor
- Assumes that a factor’s effect is unidirectional
Example

Start with a $2^2$ experimental design.

<table>
<thead>
<tr>
<th>Cache</th>
<th>Memory</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>75</td>
</tr>
</tbody>
</table>

Define two variables $x_A$ and $x_B$ as $x_A = -1$ for 4MB memory, and 1 for 16MB memory; $x_B = -1$ for 1KB cache, and 1 for 2KB cache.

Regression equation:

$$ y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B $$

For this example, we get 4 equations and 4 unknowns:

$$
\begin{align*}
15 & = q_0 - q_A - q_B + q_{AB} \\
45 & = q_0 + q_A - q_B - q_{AB} \\
25 & = q_0 - q_A + q_B - q_{AB} \\
75 & = q_0 + q_A + q_B + q_{AB}
\end{align*}
$$
Example (cont)

Solve to yield:

\[ y = 40 + 20x_A + 10x_B + 5x_Ax_B. \]

Interpretation: Mean performance is 40 MIPS, the effect of memory is 20 MIPS, the effect of cache is 10 MIPS, and the interaction effect is 5 MIPS.

Sign table:

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>45</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>75</td>
</tr>
<tr>
<td>160</td>
<td>80</td>
<td>40</td>
<td>20</td>
<td>Total</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>Tota/4</td>
</tr>
</tbody>
</table>
Allocation of Variation

The importance of a factor is measured by the proportion of the total variation in the response that is explained by the factor.

Sum of Squares Total (SST) or variation:

\[ \sum_{i=1}^{2^2} (y_i - \overline{y})^2. \]

\[ SST = SSA + SSB + SSAB \]
\[ = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}. \]

2100 = 1600 + 400 + 100.

The fraction of variation explained by A is SSA/SST.

If one factor explains the vast majority of the variation, then the other can be ignored.
- In the earlier example, memory explains 76% of variation, cache explains 19%.
**General \(2^k\) Factorial Designs**

We can generalize this to \(k\) factors.

Example: memory size, cache size, number of processors.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Level -1</th>
<th>Level 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Memory (A)</td>
<td>4MB</td>
<td>16MB</td>
</tr>
<tr>
<td>Cache (B)</td>
<td>1KB</td>
<td>2KB</td>
</tr>
<tr>
<td>Processors (C)</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>4MB</th>
<th>16MB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cache</td>
<td>One P</td>
<td>Two P</td>
</tr>
<tr>
<td></td>
<td></td>
<td>One P</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>46</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

\[SST = 2^3(q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2)\]
\[= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2)\]
\[= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512.\]

Memory explains 18\%, cache explains 4\%, and number of processors explains 71\%.
2^{kr} Factorial Designs

Experiments are observations of random variables.

With only one observation, can’t estimate error.

If we repeat each experiment \( r \) times, we get \( 2^{kr} \) observations.

With a 2 factor model, we now have:

\[
y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e.
\]
Example

\[
\begin{array}{cccccc}
y & \bar{y} & \text{errors} \\
15 & 18 & 12 & 15 & 0 & 3 & -3 \\
45 & 48 & 51 & 48 & 3 & 0 & 3 \\
25 & 28 & 19 & 24 & 1 & 4 & -5 \\
75 & 75 & 81 & 77 & -2 & -2 & 4 \\
\end{array}
\]

Coefficients (divide by 4):
\[ q_0 = 41, \ q_A = 21.5, \ q_B = 9.5, \ q_{AB} = 5 \]

\[
SST = SSA + SSB + SSAB + SSE
\]

\[ 7032 = 5547 + 1083 + 300 + 102 \]

Factor A explains 79\%, B explains 15.4\%, AB explains 4\%, and error explains about 1.5\%.

We can also compute confidence intervals. The degrees of freedom are \(2^k(r - 1)\).

- Factor A has 90\% confidence interval \(41 \pm 1.92\). So it is significant.
\[2^{k-p}\] Fractional Factorial Designs

Even with \(2^k\) designs, the number of experiments can get out of hand for several factors.

We can get a lot of information with fewer experiments.

- If we pick the experiments carefully, we can get enough information to compute what we need most.
- Do the following set of 8 experiments on 7 factors.

<table>
<thead>
<tr>
<th>#</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Selecting the Experiments

The selection of levels is important.
• The vectors need to be orthogonal so that the contributions of the effects can be determined.
• Each column sums to zero
• The sum of products of any two columns is zero
• There are $2^{k-p}$ rows

$$q_j = \sum_i y_i x_{ji}$$

That is, sum up the column values times their coefficients and divide by 8.
Example

\[
\begin{array}{cccccccc}
I & A & B & C & D & E & F & G & y \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 20 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 35 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 7 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 42 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 36 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 50 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 45 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 82 \\
317 & 101 & 35 & 109 & 43 & 1 & 47 & 3 & \text{Tot} \\
39.6 & 12.6 & 4.4 & 13.6 & 5.4 & 0.1 & 5.9 & 0.4 & \text{Tot/8} \\
\end{array}
\]

Calculated variation by variable:
\[A = 37\%, \quad B = 5\%, \quad C = 43\%, \quad D = 7\%, \quad E = 0\%, \quad F = 8\%, \quad G = 0\%\]

Further experiments should be conducted only on A and C.

Big assumption: These experiments only provide so much information.

- The effects due to interaction are in the values calculated for the separate variables
- The experiments are masking these “conflouning” interactions
- If the interaction effects are small, its OK