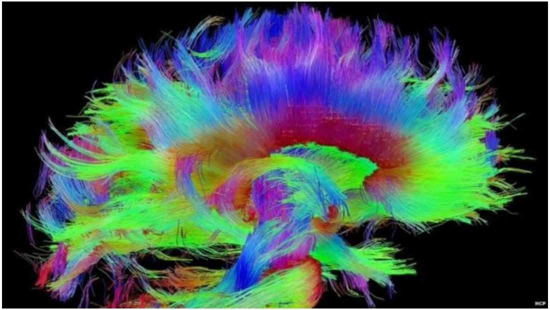
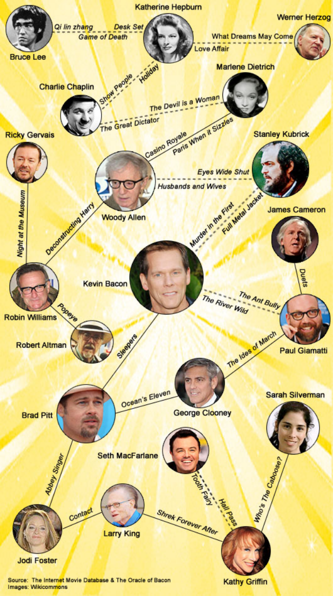
CS 4984: Introduction to Graphs

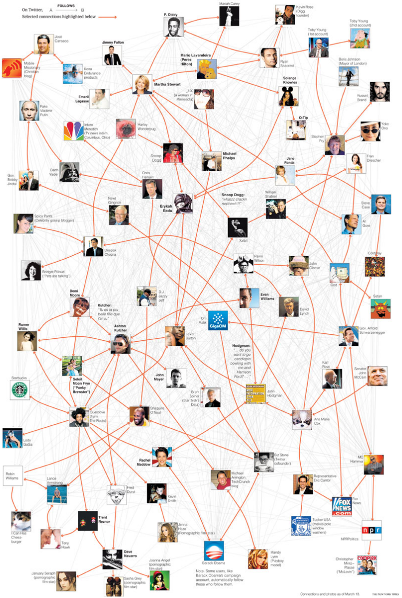
T. M. Murali

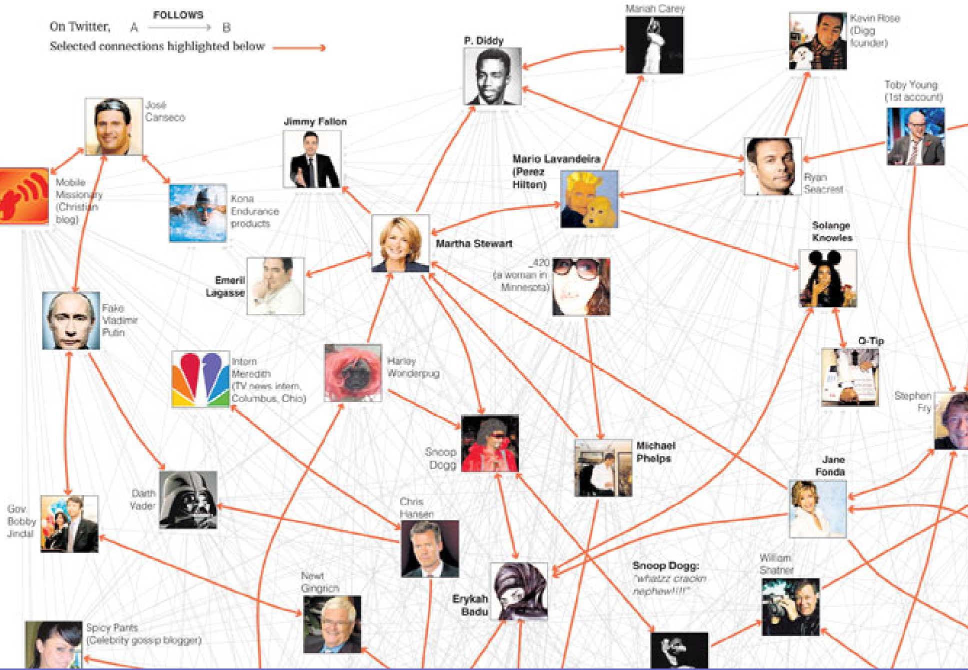
January 25, 2018

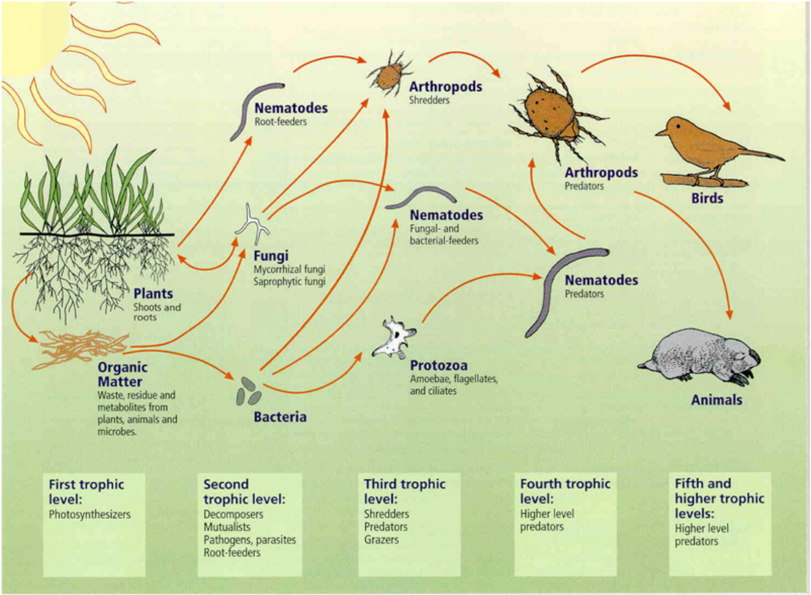


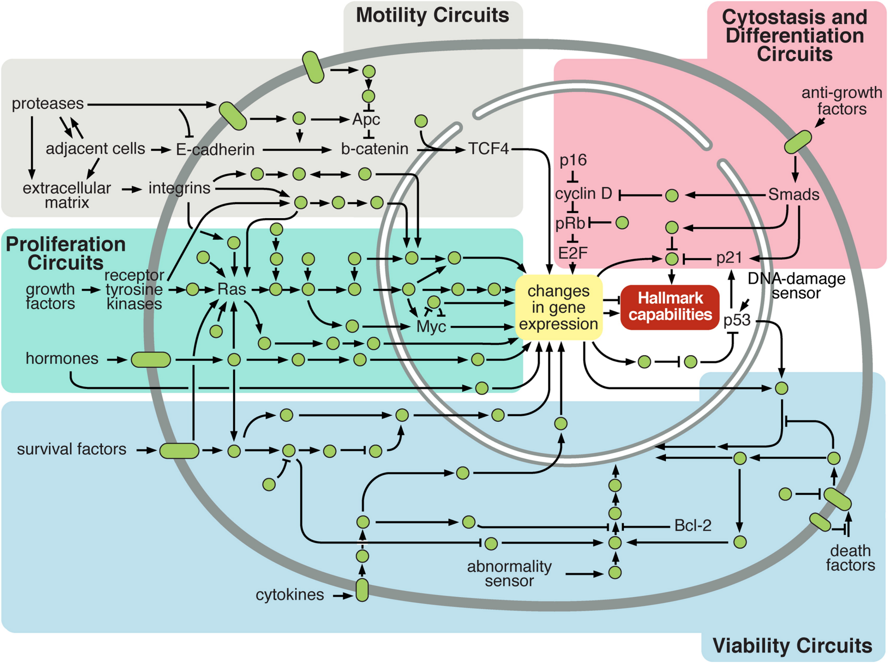


The Oracle of Bacon




On Twitter, A FOLLOWS B
Selected connections highlighted below
Mariah Carey
Kevin Rose (Digg founder)
P. Diddy
Toby Young (1st account)
José Canseco
Jimmy Fallon
Mario Lavandeira (Perez Hilton)
Mobile Missionary (Christian blog)
Kona Endurance products
Ryan Seacrest
Martha Stewart
Solange Knowles
_420 (a woman in Minnesota)
Emeril Lagasse
Fake Vladimir Putin
Q-Tip
Intern Meredith (TV news intern, Columbus, Ohio)
Harley Wonderpug
Stephen Fry
Snoop Dogg
Michael Phelps
Jane Fonda
Darth Vader
Gov Bobby Jindal
Chris Hansen
William Shatner
Snoop Dogg: "whatzz crackn nephew!!!!"
Erykah Badu
Newt Gingrich
Spicy Pants (Celebrity gossip blogger)


Arthropods
Shredders
Nematodes
Root-feeders
Birds
Arthropods
Predators
Nematodes
Fungal- and
bacterial-feeders
Fungi
Mycorrhizal fungi
Saprophytic fungi
Plants
Shoots and
roots
Nematodes
Predators
Protozoa
Amoebae, flagellates,
and ciliates
Organic
Matter
Waste, residue and
metabolites from
plants, animals and
microbes.
Bacteria
Animals
First trophic
level:
Photosynthesizers
Second
trophic level:
Decomposers
Mutualists
Pathogens, parasites
Root-feeders
Third trophic
level:
Shredders
Predators
Grazers
Fourth trophic
level:
Higher level
predators
Fifth and
higher trophic
levels:
Higher level
predators



Motility Circuits
Cytostasis and Differentiation Circuits
proteases
adjacent cells
E-cadherin
b-catenin
Apc
TCF4
extracellular matrix
integrins
anti-growth factors
Smads
p16
cyclin D
pRb
E2F
p21
Proliferation Circuits
growth factors
receptor tyrosine kinases
Ras
Myc
changes in gene expression
Hallmark capabilities
DNA-damage sensor
p53
hormones
survival factors
Bcl-2
abnormality sensor
cytokines
death factors
Viability Circuits

Graphs

Graph $\equiv$ Network

- Model pairwise relationships (edges) between objects (nodes).

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- Other examples:

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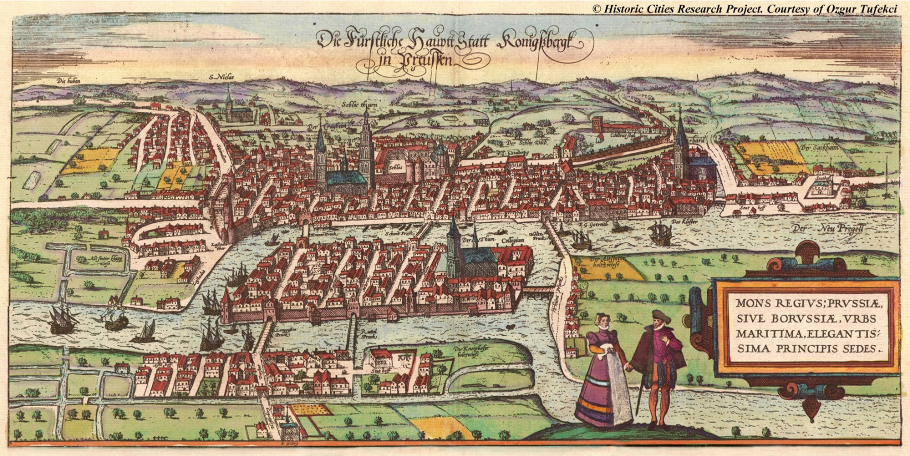
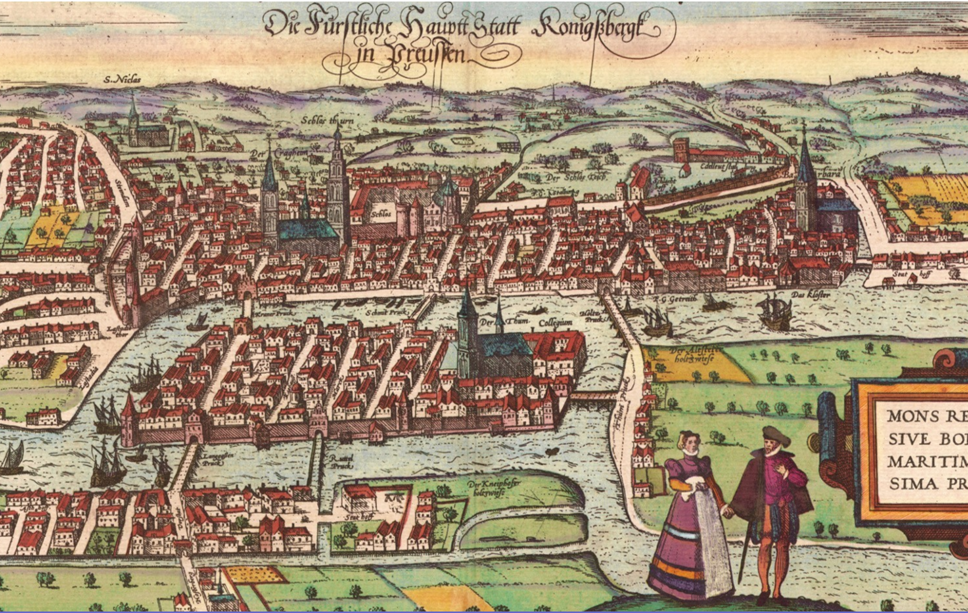
- Problems involving graphs have a rich history dating back to Euler.

Euler and Graphs



Devise a walk through the city that
crosses each of the seven bridges exactly once.

Euler and Graphs



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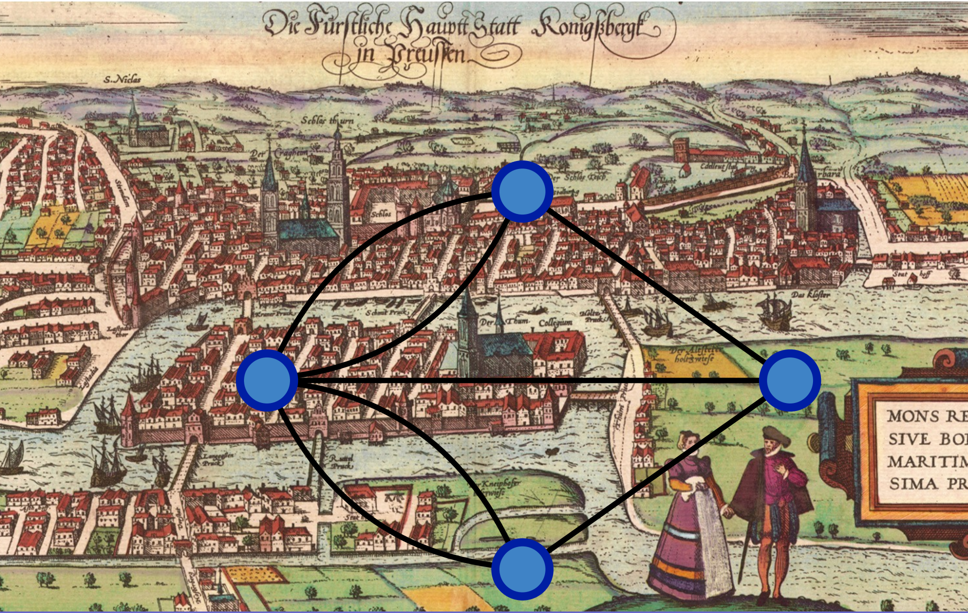
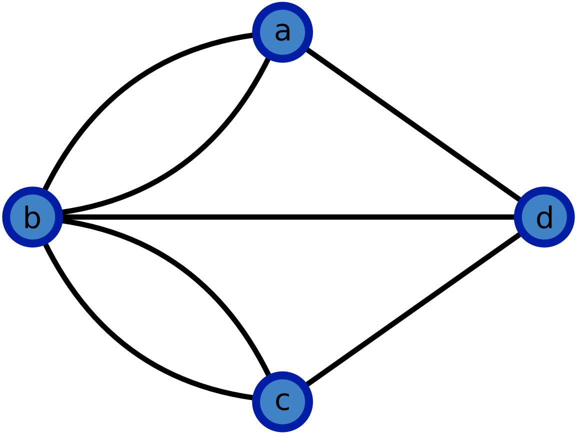


Definition of an Undirected Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges.
  - Each element of E is an unordered pair of nodes.
  - Edge (u, v) is *incident* on u, v; u and v are *neighbours* of each other.
  - G contains no self loops.



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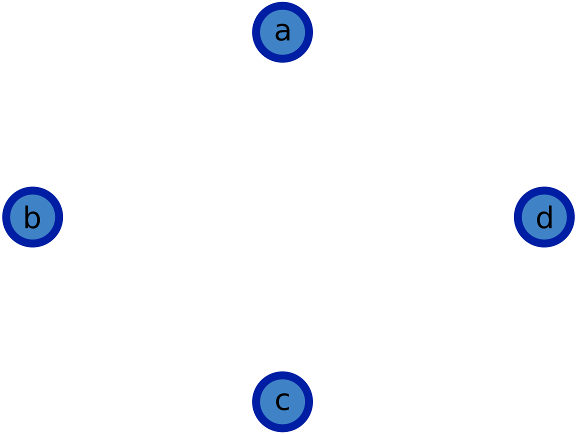

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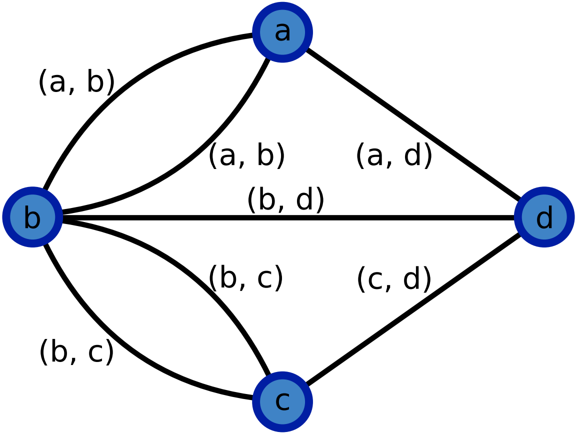
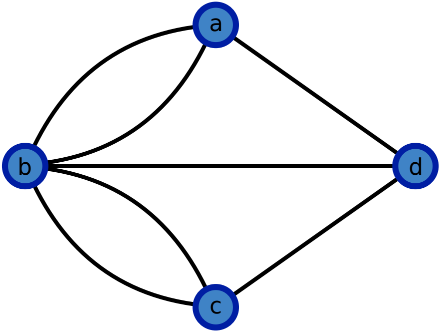


Paths and Cycles in Graphs



- A *v_1-v_k path* in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that for every $i, 1 \leq i < k$, (v_i, v_{i+1}) is an edge in E.

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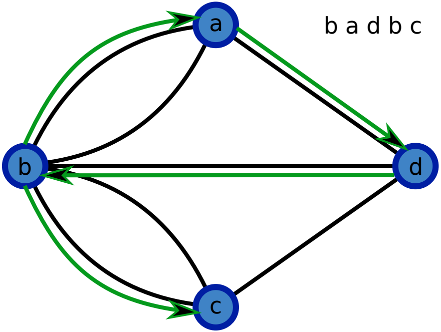

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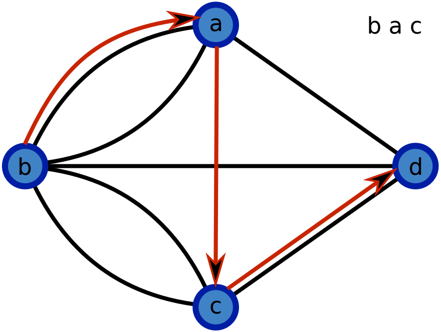

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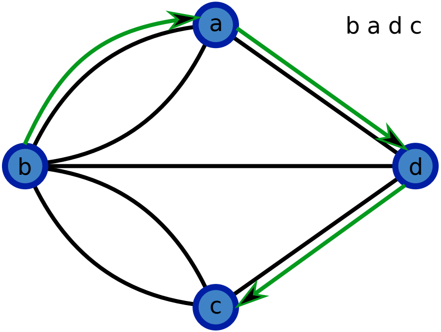

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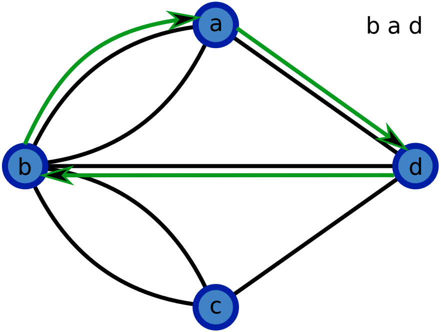

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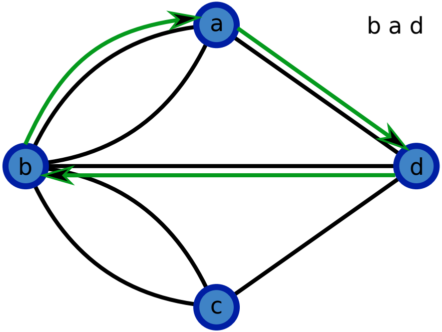

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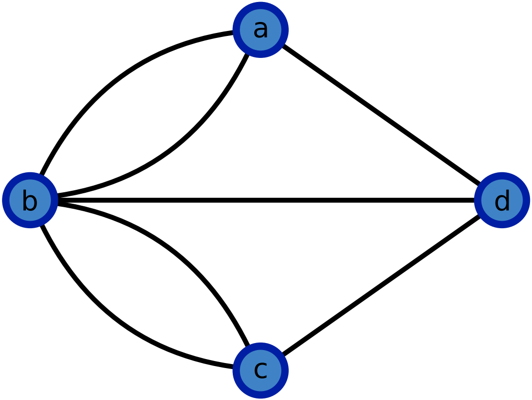
- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where the first $k-1$ nodes are distinct and $v_1 = v_k$.
- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a u-v path in G.

Bridges to Graphs



EULERIAN TOUR

Given an undirected graph $G(V, E)$,
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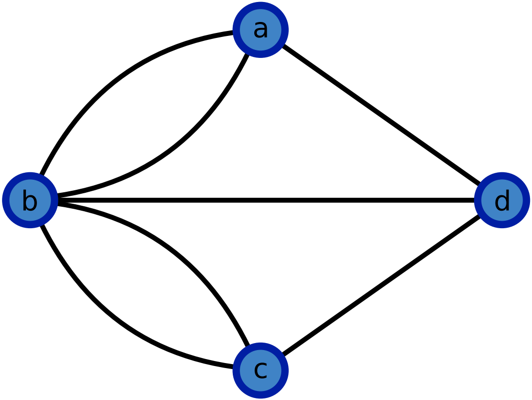


EULERIAN TOUR

Given an undirected graph $G(V, E)$,
construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

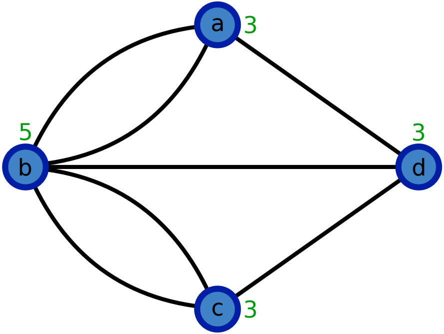
What Euler Proved

§. 19. Praeterea ſi duo tantum numeri litteris A, B, C etc. adſcripti fuerint impares, reliqui vero omnes pares, tum ſemper deſideratus tranſitus ſuccedet, ſi modo curſus ex regione ad quam pontium impar numerus tendit incipiatur. Si enim pares numeri biſecentur atque etiam impares vnitate aucti, vti praeceptum eſt, ſumma harum medietatum vnitate erit maior quam numerus pontium, ideoque aequalis ipſi numero praefixo. Ex hocque porro perſpicitur, ſi quatuor vel ſex vel octo etc. fuerint numeri impares in ſecunda columna, tum ſummam numerorum tertiae columnae maiorem fore numero praefixo, eumque excedere vel vnitate, vel binario vel ternario etc. et idcirco tranſitus fieri nequit.

§. 20. Caſu ergo quocunque propoſito ſtatim facillime poterit cognoſci, vtrum tranſitus per omnes pontes ſemel inſtitui queat an non, ope huius regulae. Si fuerint plures duabus regiones, ad quas ducentium pontium numerus eſt impar, tum certo affirmari poteſt, talem tranſitum non dari. Si autem ad duas tantum regiones ducentium pontium numerus eſt impar, tunc tranſitus fieri poterit, ſi modo curſus in altera harum regionum incipiatur. Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum tranſitus deſiderato modo inſtitui poterit, in quacunque regione ambulandi initium ponatur. Hac igitur data regula problemati propoſito pleniſſime ſatisfit.

S 2 §. 21.

What Euler Proved (in English)



- *Degree $d(v)$ of a node v* is the number of edges incident on it.

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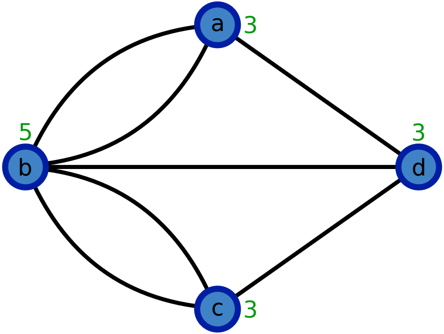

- *Degree $d(v)$ of a node v* is the number of edges incident on it.
- Euler's conclusion:
  1. If there are more than two nodes with odd degree, then the graph has no Eulerian tour.
  2. If exactly two nodes in the graph have odd degree, then there exists a tour that starts at one of these nodes and ends at the other node.
  3. If all nodes have even degree, then there exists a tour starting at any node.

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- What about constructing such a tour if it exists?

What Didn't Euler Prove?

§. 21. Quando autem inuentum fuerit talem transitum instituí posse, quaestio superest quomodo cursus sit dirigendus. Pro hoc sequenti vtor regula; tollantur cogitatione quoties fieri potest, bini pontes, qui ex vna regione in aliam ducunt, quo pacto pontium numerus vehementer plerumque diminuetur, tum quaeratur, quod facile fiet, cursus desideratus per pontes reliquos, quo inuento pontes cogitatione sublati hunc ipsum cursum non multum turbabunt, id quod paululum attendenti statim patebit; neque opus esse indico plura ad cursus reipsa formandos praecipere.

THEO-

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- Euler's conditions (1741) were necessary. Hierholzer proved their sufficiency (1873).
- What about constructing such a tour if it exists?
  - We must go through the effort to write out a path that is correct.
  - Method to accomplish this was trivial, and Euler did not want to spend a great deal of time on it.

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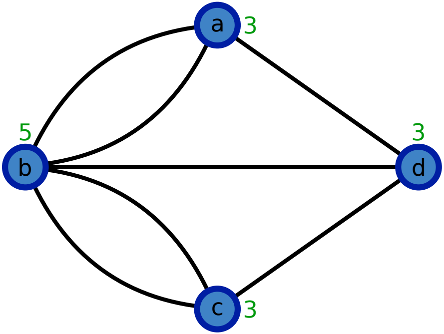
- Hierholzer provided an algorithm.

Hierholzer's Algorithm



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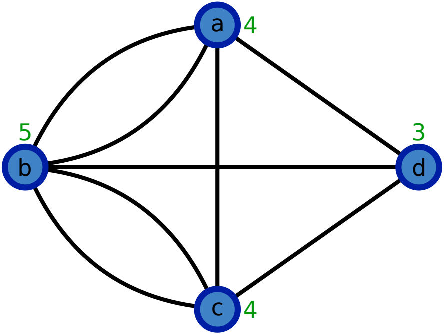

- If there are two nodes in G with odd degree, call them s and t.
- Otherwise, let s be any node in G.

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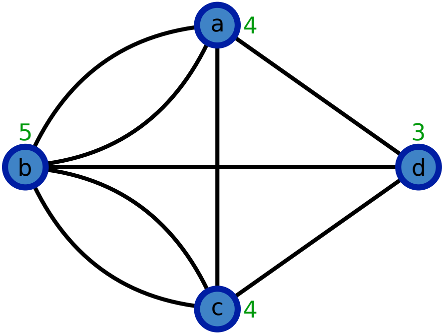

- If there are two nodes in G with odd degree, call them s and t.
- Otherwise, let s be any node in G.

$u \leftarrow s$ #u denotes the currently-visited node.
while $d(u) > 0$ **do**
  Output u.
  Let v be a neighbour of u.
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end while

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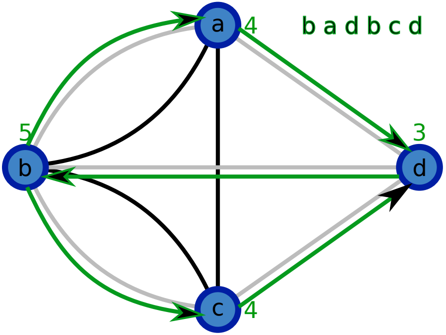

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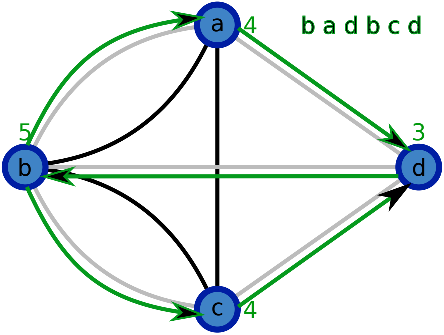

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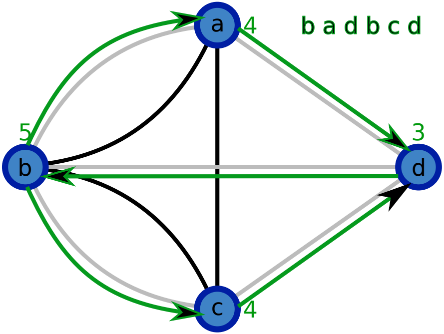

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- Will the algorithm terminate?
- If it terminates, what can we say about node u at termination?
- Will all edges of G have been traversed upon termination?

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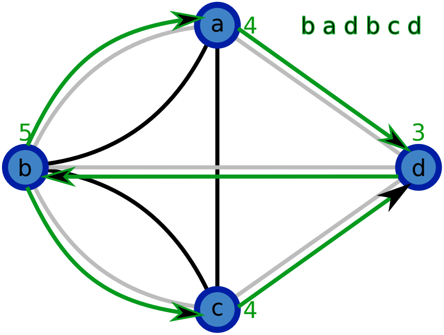

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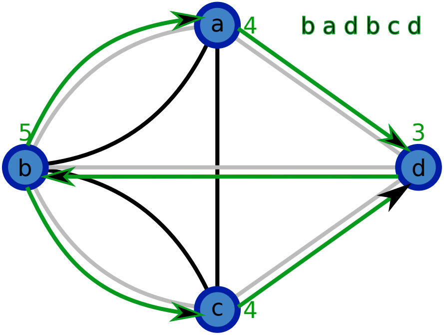

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- If it terminates, what can we say about node u at termination?
  - If G had no nodes of odd degree, then $u = s$.
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- Will all edges of G have been traversed upon termination?

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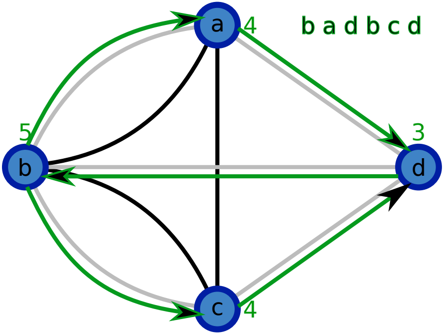

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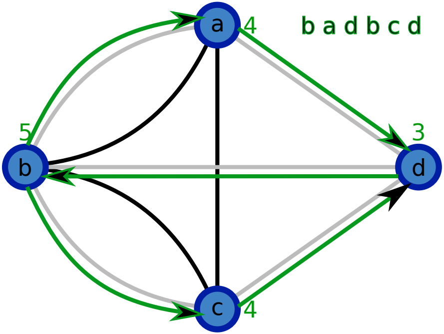

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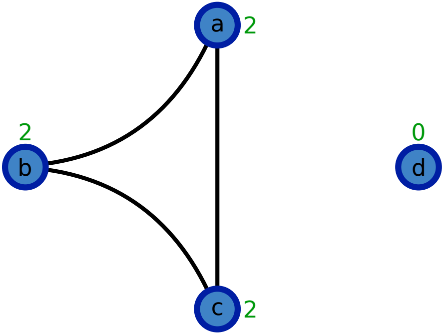

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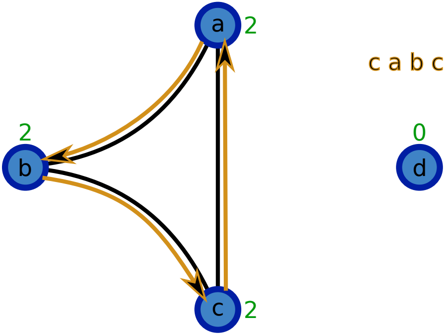
- Will all edges of G have been traversed upon termination? No! Set u to be any node on the path output so far and repeat.

Properties of Heilholzer's Algorithm



$u \leftarrow s$ #u denotes the currently-visited node.
while $d(u) > 0$ **do**
  Output u.
  Let v be a neighbour of u.
  Delete the edge (u, v) from G.
  $u \leftarrow v$
end while

- Will the algorithm terminate? Yes, because we traverse a new edge in each iteration.
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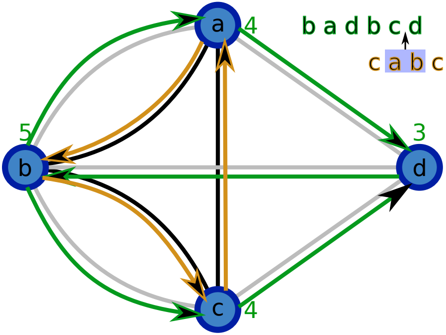

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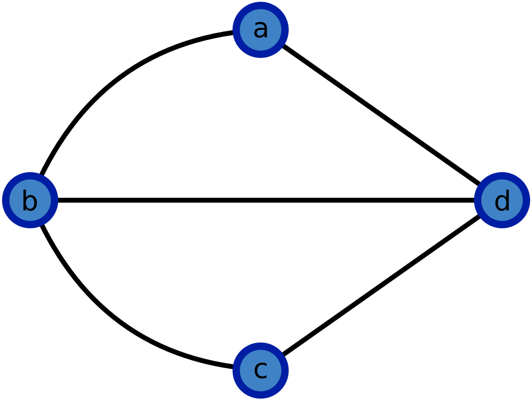
- Algorithm's running time is $O(|V| + |E|)$, i.e., linear in the size of G.

Visiting Nodes Rather than Edges



EULERIAN TOUR

Given an undirected graph $G(V, E)$,
construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

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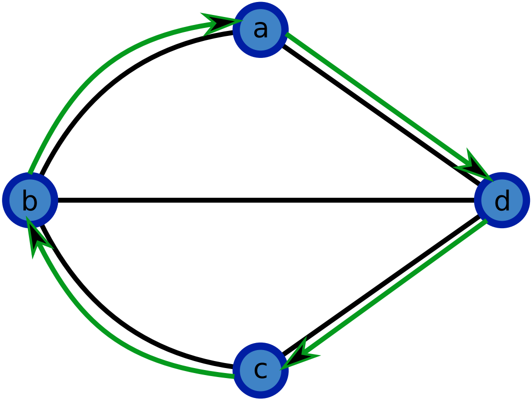

HAMILTONIAN CYCLE
Given an undirected graph $G(V, E)$,
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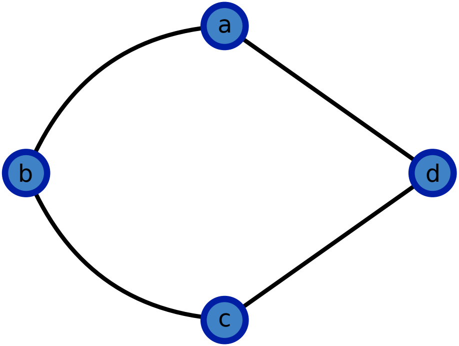
Conditions for Existence of Hamiltonian Cycle



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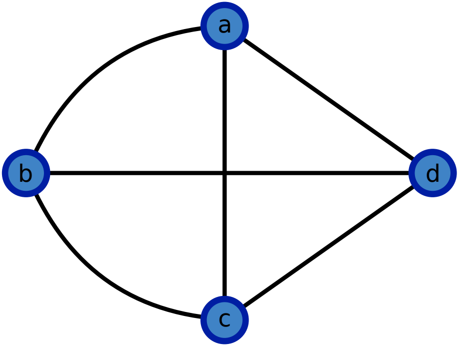

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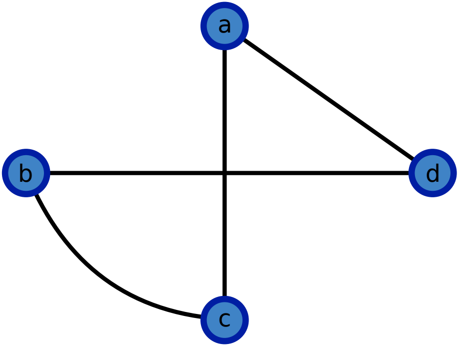

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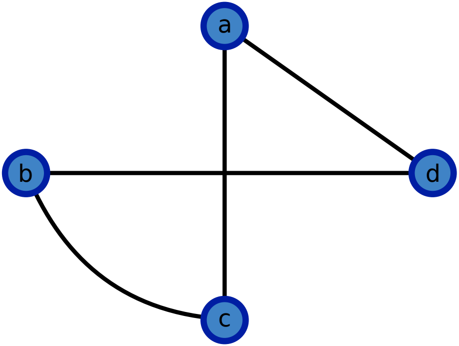

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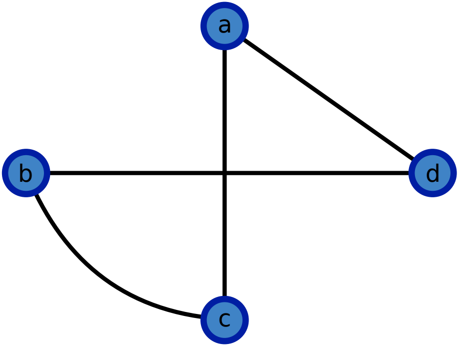

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  - two disconnected nodes with sum of degrees $\geq n$ (Ore, 1952).

Status of Hamiltonian Cycle Problem

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  - Dynamic programming: running time of $O(n^2 2^n)$ (Held and Karp 1962).
  - Fastest known algorithm runs in time $O(1.657^n)$ (Björklund 2010).