1 Problem set-up

This is a follow up on the "bug population" model discussed in class. As discussed in class, there are several ways to introduce non-trivial, oscillatory behavior into your model. An alternative to what we did would be to consider a second-order ODE, which, as you know, can always be reduced to two coupled differential equations. Biologically, this amounts to introducing an additional realism into the basic model: indeed, in real life, bugs are never left alone – birds eat them. In fact, this is how the bug population is kept in check. When birds are present, bug population $p$ never becomes large enough for the quadratic term $\sim p^2$ to matter, so you can drop it from the model, keeping only $dp/dt = \alpha p$, $\alpha > 0$. Here $t$ is the time variable. Instead, to describe the balance of the bug population, you add a term like $\beta px$, where $x$ is the bird population. The end result for the bug balance equation is $dp/dt = \alpha p + \beta px$. The meaning of the second term is that the change in the bug population is proportional to the number of bugs and the number of birds that eat the poor critters. To close the system of equations, you need to balance the birds. Assume that without food (bugs), any initial bird population simply dies off exponentially. This behavior of the bird population is described by the first term $A = A(x)$ in the bird balance $dx/dt = A + B$. To arrive at the form of $B$, it is reasonable to assume that the more bugs we have, the higher is the bird birth rate. So, $B = \delta px$.

2 Questions

Give a clear answer and provide an explanation. No explanation = no points.

Q1, 5 points  What is the sign of $\beta$?

Q1, answer  The meaning of the $\beta px$ term is the decrease of the bug population due to birds eating the bugs. Thus, $dp/dt$ due to this term alone is $< 0$. Since, by definition, both $p$ and $x$ are positive, $\beta$ must be negative.

Q2, 5 points  What is the sign of $\delta$?

Q2, answer  $B = \delta px$ is the birth rate of birds, the more bugs the higher. Birth means increase in the bird population, so $\delta > 0$.

Q3, 5 points  What is the mathematical form of $A(x)$? Comment on the sign of the prefactor (call it $\gamma$ for consistency).

Q3, answer  From the class lecture, we know that exponential decay of the population means that $x(t) = \text{const} \times \exp(\gamma t)$. Decay = decrease, so $\gamma < 0$. Thus, to describe this behavior via the differential equation $dx/dt = A(x)$ we need $A(x) = \gamma x$, $\gamma < 0$.

Q4, 5 points  Write down the full system of 2 coupled ODEs that describes the bugs-birds balance and time-evolution of the populations. The system will have 4 parameters: $\alpha, \beta, \gamma, \delta$.

Q4, answer  Putting all of the above together, we have:

$$dp/dt = \alpha p + \beta px \quad (1)$$
$$dx/dt = \gamma x + \delta px \quad (2)$$
Q5, 5 points  Find stationary solutions. What do they mean?

Q5, answer  By definition, both populations do not change with time at the stationary point. To find those, we need to solve, simultaneously,

\[
\begin{align*}
\frac{dp}{dt} &= \alpha p + \beta px = 0 \quad (3) \\
\frac{dx}{dt} &= \gamma x + \delta px = 0 \quad (4)
\end{align*}
\]

which leads to one trivial stationary point \((0,0)\) and one non-trivial: \((-\frac{\alpha}{\beta}, -\frac{\gamma}{\delta})\)

Q6, 10 points  Use Mathematica to solve (numerically) the system of equations. Show the key lines of your code. Discuss at least two “sanity checks” for your numerical solution, show the plots.

Q6, answer  

For the Mathematica script, see below. Checks: There are many options. e.g. set \(\beta = \delta = 0\) and watch the bug population explode (birds do not eat bugs) and bird’s die off. Or you can set your initial condition to the second stationary point and observe no change, at least for a while (at longer time, numerical issues may cause deviation from the stationary behavior).

Sanity Check 1: bug population explodes and bird population dies off when \(\alpha = 1, \beta = 0, \gamma = -1, \delta = 0,\) and \(p(0) = 2, x(0) = 10\) (Figure 1)

Sanity Check 2 (the stationary point): both populations stay the same when \(\alpha = 2, \beta = -1, \gamma = -2, \delta = 1,\) and \(p(0) = -\frac{\alpha}{\beta} = 2, x(0) = -\frac{\gamma}{\delta} = 2\) (Figure 2)

Figure 1: Sanity Check 1. Red line: birds; Blue line: bugs.

Figure 2: Sanity Check 2. Red line: birds; Blue line: bugs. (lines coincide)
Q7, 10 points  Show that, for some values of parameters, you can have oscillations in the system. Show the plots. Can you give a hand-waving explanation for why?

Q7, answer  Figure 3 is the script to set up the equation in Mathematica, assuming $\alpha = 2, \beta = -1, \gamma = -2, \delta = 1$, and $p(0) = 5, x(0) = 8$. The plot of $x(t), p(t)$ clearly shows oscillations.

```plaintext
In[1]:= s = NDSolve[
{x'[t] - 2 x[t] - p[t] x[t], x'[t] - 2 x[t] + p[t] x[t], y[0] - 5, x[0] - 8},
{p, x}, {t, 0, 20}]

Out[1]:= {{{p -> InterpolatingFunction[{{{0., 20.}}, <<1>>},
{x -> InterpolatingFunction[{{{0., 20.}}, <<1>>}]]}}

In[3]:= Plot[Evaluate[{p[t], x[t]} /. s], {t, 0, 20}, PlotRange -> All]
```

Figure 3: Typical oscillatory behavior in the predator-prey model. Red line: birds; Blue line: bugs.

The meaning of it is as follows. Initially, there are more birds than bugs, so bug population decreases. But this also means less food for birds, so fewer chicks, which means fewer birds. As the number of birds declines, bug population starts to increase, and when the number of bugs increases, more birds are born. At some point, the birds ”overeat” the bugs, and the bug population starts to decline again, and the cycle repeats itself. See e.g. ”Lotka–Volterra equation” on Wiki.