CS 4124

A Worked Example of the Lupanov Representation

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Here is another worked example of the Lupanov representation of a Boolean function. The Boolean function $f : \mathcal{B}^n \to \mathcal{B}$ (truth table in Figure 1) has $n = 2$ and $k = 2$. Hence, $n - k = 2$. The example is a Boolean function $f : \mathcal{B}^4 \to \mathcal{B}$. Since $k = 2$ and $n - k = 2$, we think of f as a function

$$f : \mathcal{B}^2 \times \mathcal{B}^2 \to \mathcal{B}.$$

			$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_4$	
			0	0	1	1	x_3
$\mathbf{a}_{ij}$	x_1	x_2	0	1	0	1	x_4
$\mathbf{a}_{1,1}$	0	0	0	1	0	0	A_1
$\mathbf{a}_{1,2}$	0	1	0	1	1	1	
$\mathbf{a}_{2,1}$	1	0	1	0	0	1	A_2
$\mathbf{a}_{2,2}$	1	1	0	1	0	0	

Figure 1: Truth table for the $(2, 2)$-Lupanov representation of f.

The row vectors (from $\mathcal{B}^2$) are partitioned into $p = 2$ sets A_1 and A_2. Each set has $s = 2$ row vectors. The 2 row vectors in A_i are labeled as $\mathbf{a}_{ij}$, where $j \in \{1, 2\}$. So, the 4 row vectors are $\mathbf{a}_{1,1}$, $\mathbf{a}_{1,2}$, $\mathbf{a}_{2,1}$, and $\mathbf{a}_{2,2}$. Also, we label the 4 column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_2$, and $\mathbf{b}_2$.

Recall that

$$g_i : \mathcal{B}^2 \to \mathcal{B}^2$$

is defined to be

$$g_i(\mathbf{b}) \quad = \quad (f(\mathbf{a}_{i,1}, \mathbf{b}), f(\mathbf{a}_{i,2}, \mathbf{b}))$$

where $i \in \{1, 2\}$. Figure 2 gives all the g_i values for this example.

$\mathbf{b}_t$	x_3	x_4	$g_1(\mathbf{b}_t)$		$g_2(\mathbf{b}_t)$	
$\mathbf{b}_1$	0	0	0	0	1	0
$\mathbf{b}_2$	0	1	1	1	0	1
$\mathbf{b}_3$	1	0	0	1	0	0
$\mathbf{b}_4$	1	1	0	1	1	0

Figure 2: Table of g_i values.

The *column function*

$$c_i : \mathcal{B}^2 \times \mathcal{B}^2 \to \mathcal{B}$$

is defined to be

$$c_i(\mathbf{v},\mathbf{b}) \quad = \quad \begin{cases} 1 & \text{if } g_i(\mathbf{b}) = \mathbf{v}\text{; and} \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \{1, 2\}$. Figure 3 gives the values of the column functions for f.

$\mathbf{v}$		$\mathbf{b}$			
v_1	v_2	x_3	x_4	$c_1(\mathbf{v},\mathbf{b})$	$c_2(\mathbf{v},\mathbf{b})$
0	0	0	0	1	0
0	0	0	1	0	0
0	0	1	0	0	1
0	0	1	1	0	0
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	1	0
0	1	1	1	1	0
1	0	0	0	0	1
1	0	0	1	0	0
1	0	1	0	0	0
1	0	1	1	0	1
1	1	0	0	0	0
1	1	0	1	1	0
1	1	1	0	0	0
1	1	1	1	0	0

Figure 3: Table of c_i values.

The *row function*

$$r_i : \mathcal{B}^2 \times \mathcal{B}^2 \rightarrow \mathcal{B}$$

is defined to be

$$r_i(\mathbf{v},\mathbf{a}) \quad = \quad \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{a}_{ij} \text{ and } \pi_j(\mathbf{v}) = 1\text{; and} \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \{1, 2\}$. Figure 4 gives the values of the row functions for f.

The $(2, 2)$-Lupanov representation of f is

$$\begin{aligned} f(\mathbf{a},\mathbf{b}) &= \vee_{i=1}^{p} \vee_{\mathbf{v}\in\mathcal{B}^s} (r_i(\mathbf{v},\mathbf{a}) \wedge c_i(\mathbf{v},\mathbf{b})) \\ &= \vee_{i\in\{1,2\}} \vee_{\mathbf{v}\in\{(0,0),(0,1),(1,0),(1,1)\}} (r_i(\mathbf{v},\mathbf{a}) \wedge c_i(\mathbf{v},\mathbf{b})) \\ &= (r_1((0,0),\mathbf{a}) \wedge c_1((0,0),\mathbf{b})) \vee (r_1((0,1),\mathbf{a}) \wedge c_1((0,1),\mathbf{b})) \vee \\ &\quad\; (r_1((1,0),\mathbf{a}) \wedge c_1((1,0),\mathbf{b})) \vee (r_1((1,1),\mathbf{a}) \wedge c_1((1,1),\mathbf{b})) \vee \\ &\quad\; (r_2((0,0),\mathbf{a}) \wedge c_2((0,0),\mathbf{b})) \vee (r_2((0,1),\mathbf{a}) \wedge c_2((0,1),\mathbf{b})) \vee \\ &\quad\; (r_2((1,0),\mathbf{a}) \wedge c_2((1,0),\mathbf{b})) \vee (r_2((1,1),\mathbf{a}) \wedge c_2((1,1),\mathbf{b}))\,, \end{aligned}$$

$\mathbf{v}$		$\mathbf{a}$			
v_1	v_2	x_1	x_2	$r_1(\mathbf{v},\mathbf{a})$	$r_2(\mathbf{v},\mathbf{a})$
0	0	0	0	0	0
0	0	0	1	0	0
0	0	1	0	0	0
0	0	1	1	0	0
0	1	0	0	0	0
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	0	1
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	1	0
1	1	0	1	1	0
1	1	1	0	0	1
1	1	1	1	0	1

Figure 4: Table of r_i values.

where the row and column functions are as given in Figures 3 and 4.

A circuit for the $(2,2)$-Lupanov representation can be obtained either as in the earlier handout or, since this is such a small example, directly from the truth tables of the row and column functions. For example, SOPE representations of c_1, c_2, r_1, and r_2 should be fairly compact and result in a reasonable circuit.