CONVERGENCE OF POLYNOMIAL INTERPOLANTS

Let \( f \) be a continuous function on \([a, b]\), and let \( P_n(x) \) be the polynomial of degree \( \leq n \) which interpolates \( f \) at \( n + 1 \) distinct points \( x_0^{(n)}, \ldots, x_n^{(n)} \) of \([a, b]\). It is reasonable to ask whether or not \( P_n(x) \) converges to \( f(x) \) \( \forall x \in [a, b] \) as \( n \to \infty \) and the points \( x_0^{(n)}, \ldots, x_n^{(n)} \) become dense in \([a, b]\). Bernstein in 1912 proved that the polynomials interpolating \( f(x) = |x| \) at equally spaced points in \([-1, 1]\) converge to \( f(x) \) only at \(-1, 0, \) and \(1\). Runge in 1901 proved that the interpolating polynomials to \( f(x) = 1/(1 + 25x^2) \) at equally spaced points in \([-1, 1]\) diverge, which is even more surprising since \( 1/(1 + 25x^2) \) has infinitely many derivatives. However, Bernstein also proved that if the points

\[
x_k^{(n)} = \cos \frac{2k + 1}{2n + 2} \pi
\]

are used, then the interpolating polynomials for both \( |x| \) and \( 1/(1 + 25x^2) \) converge uniformly to the function. Thus the difficulty seemed to be with equally spaced points. In 1914 Faber shocked everyone by proving that for any sequence of interpolation points, there exists a continuous function for which the interpolation polynomials at those points diverge.

The convergence of \( P_n(x) \) to \( f(x) \) has to do with numbers \( \lambda_n \) (called Lebesgue constants) and numbers \( E_n(f) \) (degree of approximation). \( E_n(f) = \inf \|f - P\|_\infty \) taken over all polynomials \( P \) of degree \( \leq n \). In the 1920’s, Jackson proved a series of theorems about \( E_n(f) \). A corollary of the Jackson Theorems is that \( E_n(f) = O(1/n) \) if \( f \) satisfies a Lipschitz condition

\[
|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in [a, b].
\]

Given points \( x_0^{(n)}, \ldots, x_n^{(n)} \), let

\[
L_{n,i}(x) = \prod_{\substack{j=0 \atop j \neq i}}^{n} \frac{x - x_j^{(n)}}{x_i^{(n)} - x_j^{(n)}},
\]

be the Lagrange polynomials. Then

\[
\lambda_n(x) = \sum_{i=0}^{n} |L_{n,i}(x)|, \quad \text{and} \quad \lambda_n = \max_{[a,b]} |\lambda_n(x)|
\]

defines the Lebesgue constants. Note that they depend on the interval \([a, b]\) and the interpolation points, and have nothing to do with the function \( f \). \( E_n(f) \) depends only on \([a, b]\) and \( f \).
**Theorem 1.** Let $f \in C[a,b]$ and $P_n$ be the polynomial interpolating at $x_0^{(n)}, \ldots, x_n^{(n)} \in [a,b]$. If $x \in [a,b]$ and $\lim_{n \to \infty} \lambda_n(x) E_n(f) = 0$, then $P_n(x) \to f(x)$. If $\lambda_n E_n(f) \to 0$, then $\|f - P_n\|_\infty \to 0$.

**Proof.** Let $Q(x)$ be the polynomial of degree $\leq n$ of best approximation to $f$, so $|f(x) - Q(x)| \leq E_n(f)$.

$$P_n(x) = \sum_{i=0}^{n} f(x_i^{(n)}) \ L_{n,i}(x)$$

and

$$Q(x) = \sum_{i=0}^{n} Q(x_i^{(n)}) \ L_{n,i}(x).$$

Then

$$|f(x) - P_n(x)| \leq |f(x) - Q(x)| + |Q(x) - P_n(x)|$$

$$\leq E_n(f) + \sum_{i=0}^{n} |Q(x_i^{(n)}) - f(x_i^{(n)})| \ |L_{n,i}(x)|$$

$$\leq E_n(f) + E_n(f) \sum_{i=0}^{n} |L_{n,i}(x)|$$

$$= E_n(f) (1 + \lambda_n(x))$$

$$\leq E_n(f)(1 + \lambda_n).$$

Since $E_n(f) \to 0$ by the Weierstrass theorem, the conclusions follow. Q.E.D.

**Theorem 2 (Bernstein).** For the Chebyshev points $x_k^{(n)} = \cos \frac{2k+1}{2n+2} \pi$ on $[-1,1]$, $\lambda_n = \mathcal{O}(\log n)$.

**Theorem 3.** Let $f$ satisfy a Lipschitz condition on $[-1,1]$ and $x_k^{(n)}$ be the Chebyshev points. Then the interpolating polynomials $P_n(x)$ converge to $f(x)$ uniformly on $[-1,1]$.

**Proof.** By the Jackson Theorem, $E_n(f) = \mathcal{O}(1/n)$. By Theorem 2, $\lambda_n = \mathcal{O}(\log n)$. Therefore

$$\lambda_n E_n(f) = \mathcal{O}(\log n) \mathcal{O}(1/n) = \mathcal{O} \left( \frac{\log n}{n} \right) \to 0.$$

Therefore by Theorem 1, $\|f - P_n\|_\infty \to 0$. Q.E.D.

This explains the convergence of the interpolating polynomials at the Chebyshev points for $F(x) = |x|$ and $f(x) = 1/(1 + 25x^2)$.

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**CS/MATH 3414 Homework # 5**

(3) 1. Let $F(x) = |x|$, $x_j = -1 + 2j/n$, $j = 0, \ldots, n$. Compute the polynomial $P_n$ interpolating $F$ at the $x_j$ for $n = 6, 8, 10, 12, 14$. On $[-1,1]$ graph the polynomials $P_n$ and $F$ (using Mathematica). (This is a variant of Problem 4.2.9 in Cheney and Kincaid.)

(3) 2. Same as 1. except use the Chebyshev points $x_j = \cos \frac{2j+1}{2n+2} \pi$, $j = 0, \ldots, n$.

(10) 3. Problems 34, 35, 36, 37, 38, page 166, in Cheney and Kincaid.

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**Extra Credit**

(3) 4. The problem here is to compute the Hermite interpolating polynomial. For any $f \in C[-1,1]$, Féjer proved that the Hermite polynomial $P_n(x)$ which satisfies $P_n(x_k) = f(x_k)$, $P_n'(x_k) = 0$ at the Chebyshev points converges uniformly to $f$ on $[-1,1]$ as $n \to \infty$. Verify this by graphing $P_n(x)$ and $f(x)$ on $[-1,1]$ for $n = 4, 6, 8, 10, 12$ and $f(x) = |x|$.
(12) 5. Let \( P(x) = \sum_{k=0}^{n} a_k \prod_{i=0}^{k-1} (x - x_i) \) and consider the divided difference table below with \( n + 2 \) distinct points \( z, x_0, \ldots, x_n \):

| \( z \) | \( b_0 \) | \( b_1 \) | \( b_2 \) | \( \cdots \) | \( b_{n-1} \) |
| \( x_0 \) | \( a_0 \) | \( \cdots \) | \( a_1 \) | \( \cdots \) | \( a_{n-1} \) |
| \( x_1 \) | \( P(x_1) \) | \( \cdots \) | \( a_2 \) | \( \cdots \) | \( a_n \) |
| \( x_2 \) | \( P(x_2) \) | \( \cdots \) | \( b_n \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( x_{n-1} \) | \( P(x_{n-1}) \) |
| \( x_n \) | \( P(x_n) \) |

1) If \( b_0 = P(z) \), write the relationship between the \( a \)'s and \( b \)'s in this divided difference table.

2) Given any point \( z \), show that an algorithm to evaluate \( P(z) \) is

\[
\tilde{b}_n := a_n; \\
\text{for } k := n - 1 \text{ step } -1 \text{ until } 0 \text{ do} \\
\tilde{b}_k := \tilde{b}_{k+1} \ast (z - x_k) + a_k; \\
\text{comment } \tilde{b}_0 = P(z);
\]

3) Using polynomial division, show that

\[
\frac{P(x)}{x - z} = \tilde{b}_1 + \tilde{b}_2 (x - x_0) + \cdots + \tilde{b}_n (x - x_0) \cdots (x - x_{n-2}) + \frac{\tilde{b}_0}{x - z}
\]

and write the relationship between the \( a \)'s and \( \tilde{b} \)'s. Conclude that

\[
b_0 = \tilde{b}_0 = \tilde{b}_0 = P(z), \quad a_n = b_n = \tilde{b}_n = \tilde{b}_n, \quad \text{and } b_i = \tilde{b}_i = \tilde{b}_i \text{ for } i = 0, \ldots, n
\]

and that

\[
P(x) = b_0 + b_1 (x - z) + b_2 (x - z) (x - x_0) + \cdots + b_n (x - z) (x - x_0) \cdots (x - x_{n-2}).
\]

4) Use the algorithm in 2) to express the Newton polynomial interpolating the data \( f(0) = 1, f(1) = 1, f'(1) = 4, f''(1) = 20, f(2) = 31, f(-1) = 1 \) in the Newton form

\[
P(x) = \sum_{j=0}^{5} d_j x^j.
\]